

A quantum categorification of the Alexander polynomial

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Winter Braids 9

- ▶ The \mathfrak{gl}_1 link invariant P_1 .

$$qP_1 \left(\text{crossing} \right) - q^{-1}P_1 \left(\text{crossing} \right) = (q - q^{-1})P_1 \left(\text{two loops} \right)$$

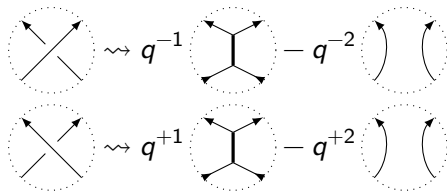
$$L \mapsto 1 \in \mathbb{Z}[q, q^{-1}]$$

- ▶ The Alexander polynomial Δ .

$$\Delta \left(\text{crossing} \right) - \Delta \left(\text{crossing} \right) = (q - q^{-1})\Delta \left(\text{two loops} \right)$$

\mathfrak{gl}_1 invariant

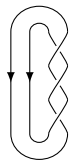
Link diagram $\rightsquigarrow \mathbb{Z}[q, q^{-1}]$ -lin. comb. of plane graphs



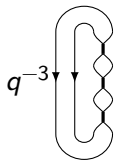
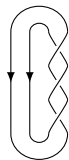
plane graph \rightsquigarrow element of $\mathbb{N}[q, q^{-1}]$

$$\Gamma \rightsquigarrow (q + q^{-1})^{\#V(\Gamma)/2} = [2]^{\#V(\Gamma)/2}.$$

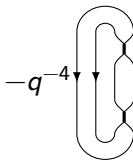
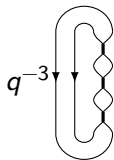
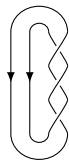
gl_1 invariant – Example



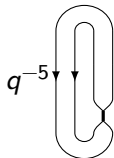
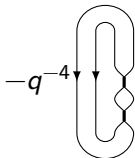
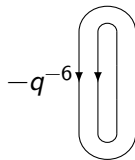
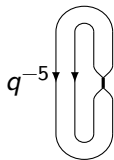
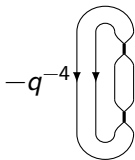
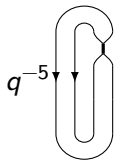
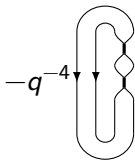
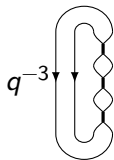
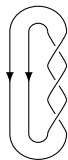
gl_1 invariant – Example



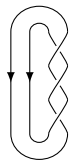
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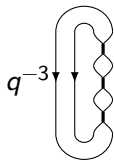
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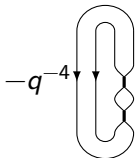
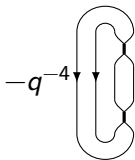
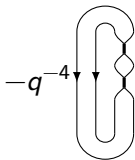
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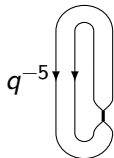
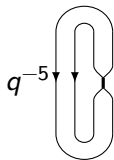
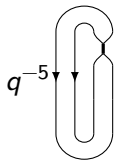
1 =



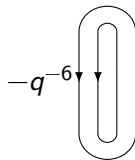
$q^{-3}[2]^3$



$-3q^{-4}[2]^2$



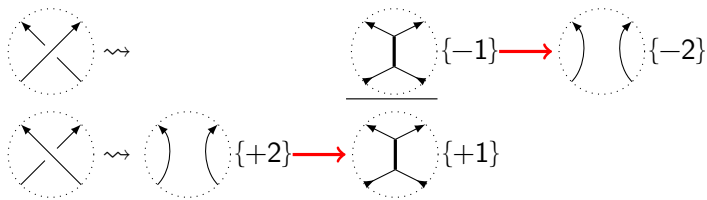
$+3q^{-5}[2]$



$-q^{-6}$

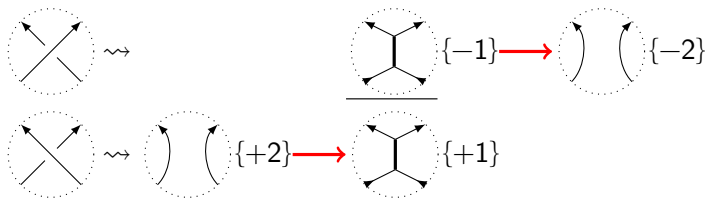
\mathfrak{gl}_1 -homology

Braid closure diagram \rightsquigarrow hypercube of plane graphs graphs (with shifts)



\mathfrak{gl}_1 -homology

Braid closure diagram \rightsquigarrow hypercube of plane graphs graphs (with shifts)

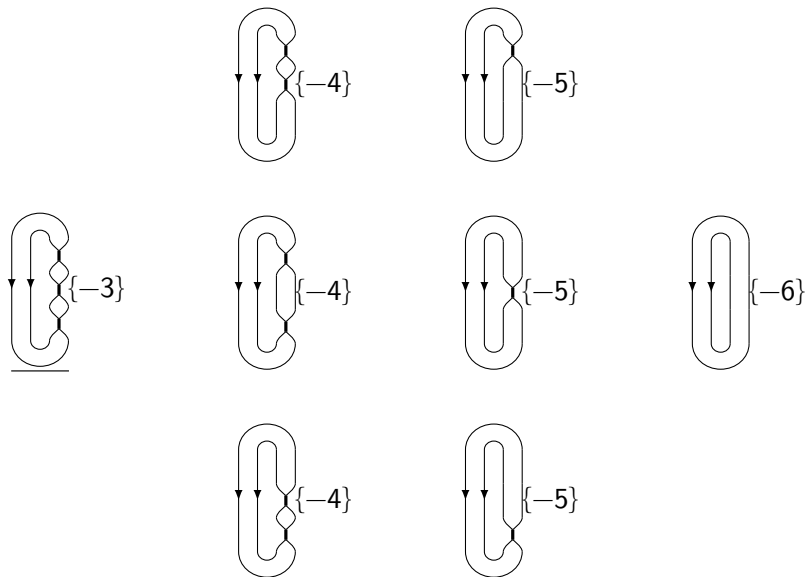


Planar (vinyl) graph \rightsquigarrow graded vector space

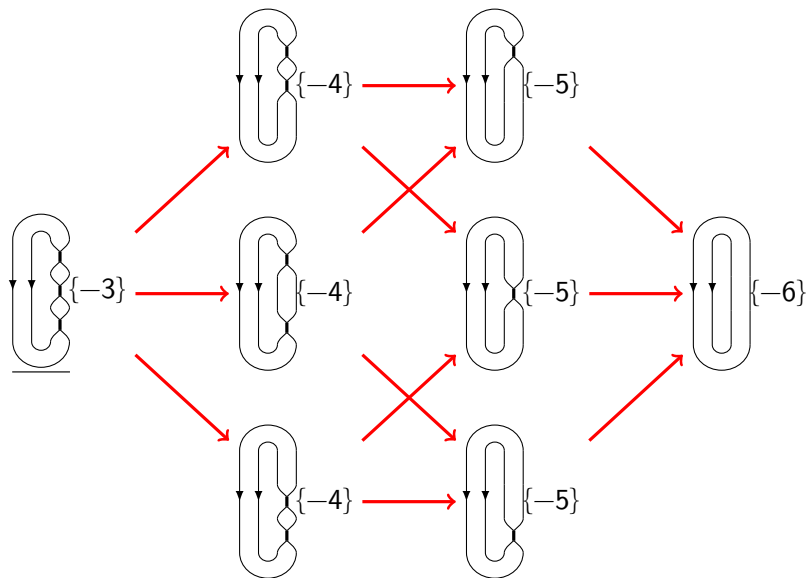
dimension $[2]^{\#\mathcal{V}(\Gamma)/2}$

\longrightarrow \rightsquigarrow graded linear map

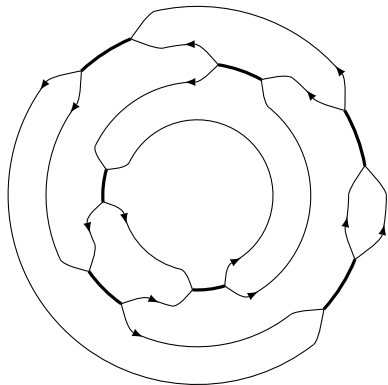
gl_1 homology – Example



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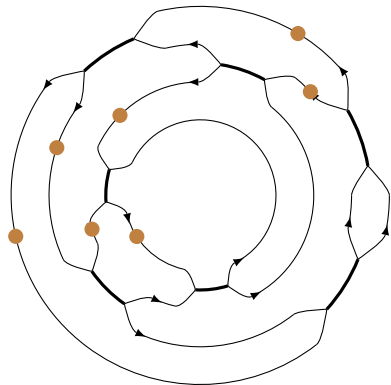


Vinyl graph \rightsquigarrow vector space



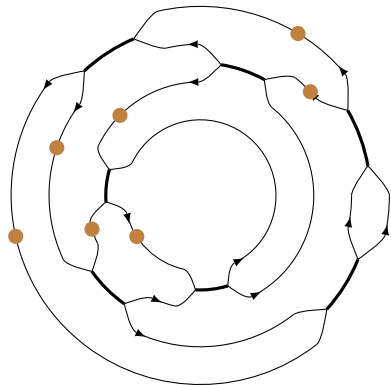
Vinyl graph Γ \circlearrowleft index k .

Vinyl graph \rightsquigarrow vector space



Vinyl graph Γ \circlearrowright index k .
Dot configuration d \bullet .

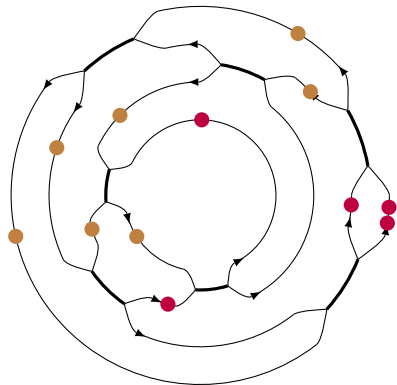
Vinyl graph \rightsquigarrow vector space



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$$D(\Gamma) = \bigoplus_d \mathbb{Q}.$$

Vinyl graph \rightsquigarrow vector space

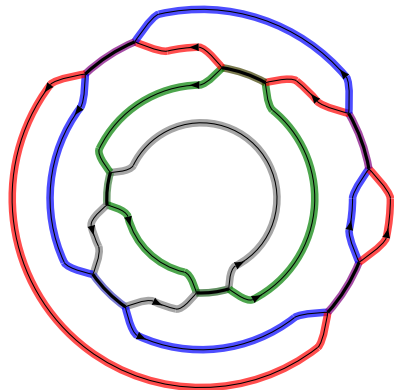


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Multiplication μ on $D(\Gamma)$.

Vinyl graph \rightsquigarrow vector space

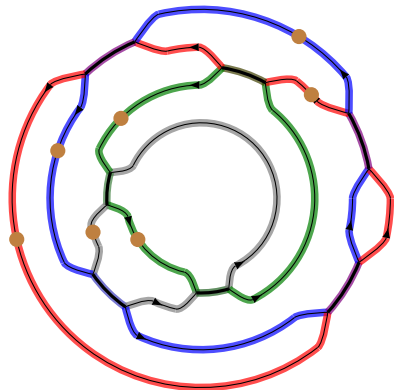


Vinyl graph Γ \circlearrowright index k .
Dot configuration d \bullet .

$$D(\Gamma) = \bigoplus_d \mathbb{Q}.$$

Multiplication μ on $D(\Gamma)$.
Coloring $c = (C_1, C_2, \dots, C_k)$
 $\Gamma = \bigsqcup_i C_i$.

Vinyl graph \rightsquigarrow vector space



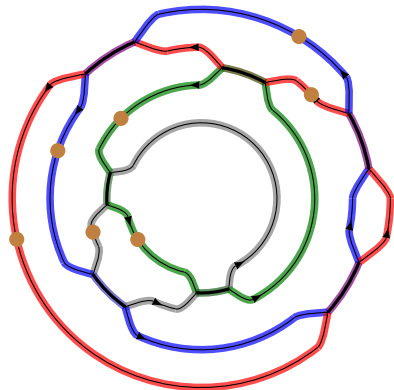
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$$\tau(d, c) = \frac{\prod_{i=1}^k X_i^{\#\{\bullet \text{ in } C_i\}}}{\prod_{\substack{C_i, C_j \\ \text{Y}}} (X_i - X_j)}$$

Vinyl graph \rightsquigarrow vector space



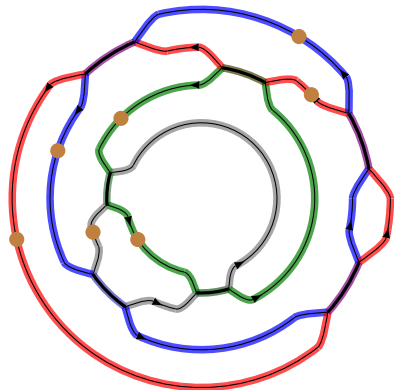
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Vinyl graph \rightsquigarrow vector space



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$$\tau(d) = \sum_{c \in \text{col}(\Gamma)} \tau(d, c)$$

Vinyl graph \rightsquigarrow vector space

Proposition (R.-Wagner, '17)

For any dot configuration d , $\tau(d) \in \mathbb{Q}[X_1, \dots, X_k]^{S_k}$.

$$\mathcal{S}_1(\Gamma) = D(\Gamma) / \ker(\tau \circ \mu_{(_, _)_{X_\bullet \mapsto 0}}).$$

Theorem (R.-Wagner, '18)

For any vinyl graph Γ , $\dim_q \mathcal{S}_1(\Gamma) = [2]^{\#V(\Gamma)/2}$.

$\longrightarrow \rightsquigarrow$ linear map

$$\longrightarrow: \mathcal{S}_1 \left(\begin{array}{c} \text{Y-junction with 3 outgoing arrows} \\ \text{Y-junction with 3 incoming arrows} \end{array} \right) \rightarrow \mathcal{S}_1 \left(\begin{array}{c} \text{Two vertical lines with arrows pointing up} \\ \text{Two vertical lines with arrows pointing down} \end{array} \right)$$
$$\mapsto$$

$$\longrightarrow: \mathcal{S}_1 \left(\begin{array}{c} \text{Two vertical lines with arrows pointing up} \\ \text{Two vertical lines with arrows pointing down} \end{array} \right) \rightarrow \mathcal{S}_1 \left(\begin{array}{c} \text{Y-junction with 3 outgoing arrows} \\ \text{Y-junction with 3 incoming arrows} \end{array} \right)$$
$$\mapsto \begin{array}{c} \text{Y-junction with 3 outgoing arrows} \\ \text{Y-junction with 3 incoming arrows} \end{array} - \begin{array}{c} \text{Y-junction with 3 outgoing arrows} \\ \text{Y-junction with 3 incoming arrows} \end{array}$$

→ \rightsquigarrow linear map

$$\begin{array}{ccc} \rightarrow: & \mathcal{S}_1 \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) & \rightarrow \mathcal{S}_1 \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) \\ & \mapsto & \mapsto \end{array} \quad \begin{array}{ccc} \rightarrow: & \mathcal{S}_1 \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) & \rightarrow \mathcal{S}_1 \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) \\ & \mapsto & \mapsto \end{array}$$

Theorem (R.-Wagner '18)

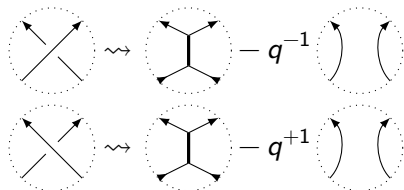
1. *These maps in the flattening of the hypercube produces a chain complex. Its homology, denoted $H_{\mathfrak{gl}_1}$ is a link invariant which categorifies P_1 .*
2. *There is a spectral sequence from the triply graded homology to $H_{\mathfrak{gl}_1}$.*

Examples

1. Trefoil: the Poincaré polynomial is $1 + q^{-4}(t + t^2)$.
2. Hopf link: the Poincaré polynomial is $1 + q^2(1 + t)$.

Alexander polynomial

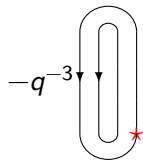
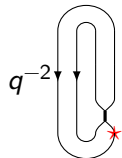
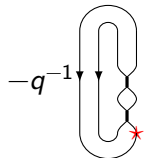
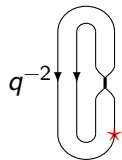
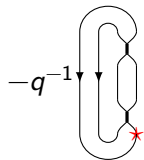
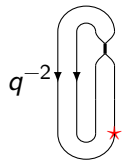
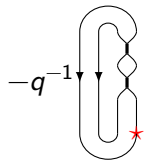
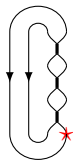
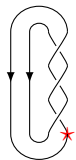
Marked (\star) braid closure $\rightsquigarrow \mathbb{Z}[q, q^{-1}]$ -lin. comb. of marked plane graphs



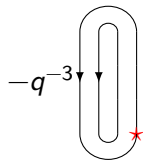
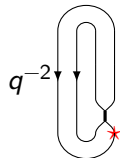
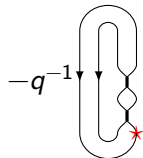
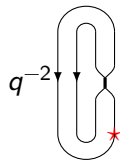
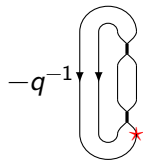
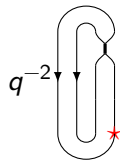
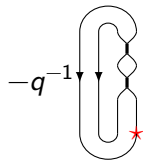
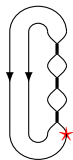
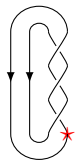
Marked plane graph \rightsquigarrow element of $\mathbb{N}[q, q^{-1}]$

$\Gamma \rightsquigarrow$ complicated (comes from $U_q(\mathfrak{gl}(1|1)) - \text{mod}$).

Alexander polynomial – Example



Alexander polynomial – Example



$$q^2 - 1 + q^{-2} =$$

$$[2]^2$$

$$-3q^{-1}[2]$$

$$+3q^{-2}$$

$$0$$

Same hypercube with a different functor.

Same hypercube with a different functor.

$$\mathcal{S}'_0(\Gamma_\star) \subseteq \mathcal{S}_1(\Gamma) = \langle \text{at least } k-1 \bullet \text{ at } \star \rangle \{-k+1\}$$

$\longrightarrow \rightsquigarrow$ induced by \mathcal{S}_1 .

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$\longrightarrow \rightsquigarrow$ induced by \mathcal{S}_1 .

Theorem (R.-Wagner, '19)

For any right-marked vinyl graph Γ_\star , $\dim_q \mathcal{S}'_0(\Gamma_\star)$ is the expected graded dimension.

Theorem (R.-Wagner '19)

1. *The flattening of the hypercube with S'_0 produces a chain complex. Its homology, denoted $H_{\mathfrak{gl}_0}$ is a knot invariant which categorifies the Alexander polynomial.*
2. *There is a spectral sequence from the reduced triply graded homology to $H_{\mathfrak{gl}_0}$.*

Thank you !!