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présentée par

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Sur l'homologie $\mathfrak{sl}_3$ des enchevêtrements ; algèbres de Khovanov – Kuperberg

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dirigée par Christian BLANCHET

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À ma sœur, à ma famille.

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Leider läßt sich eine wahrhafte Dankbarkeit mit Worten nicht ausdrücken.
Johann Wolfgang von Goethe

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[1] Celle-ci devrait se dissiper tout à fait avec la sortie de [MR13].

Résumé

Cette thèse est consacrée aux algèbres de Khovanov-Kuperberg K^ε et à leurs catégories de modules. Ce sont les analogues dans le cas $\mathfrak{sl}_3$, des algèbres H^n utilisées par Khovanov pour étendre l'homologie $\mathfrak{sl}_2$ (ou homologie de Khovanov) aux enchevêtrements. Elles apparaissent comme les images des 0-objets par une (0+1+1)-TQFT. Elles permettent de définir une homologie $\mathfrak{sl}_3$ aux enchevêtrements.

Ces algèbres ont des catégories de modules particulièrement intéressantes : du fait même de leurs constructions, elles sont profondément liées à l'étude des bases de certaines représentations du groupe quantique $U_q(\mathfrak{sl}_3)$. Il est alors naturel de vouloir classifier les modules projectifs indécomposables sur ces algèbres.

Nous étudions les modules de toiles qui sont des K^ε-modules projectifs. Il a été conjecturé que ces modules formaient une famille complète de représentants des classes d'isomorphismes de K^ε-modules projectifs indécomposables, mais Khovanov et Kuperberg ont exhibé un module de toile qui se décompose. Dans cette thèse nous donnons deux conditions sur l'indécomposabilité des modules de toiles : une condition suffisante de nature géométrique, et une condition nécessaire et suffisante de nature algébrique.

Les résultats sont prouvés d'une part grâce à une étude combinatoire des toiles, qui sont des graphes trivalents bipartites plan et d'un polynôme de Laurent qui leur est associé, le crochet de Kuperberg. Et d'autre part, grâce à l'étude des mousses qui jouent le rôle de cobordismes pour les toiles.

Mots clés : Algèbres de Khovanov-Kuperberg, Homologie $\mathfrak{sl}_3$, Crochet de Kuperberg, Toiles, Mousses, TQFT, Coloriages, Enchevêtrements.

Abstract

This thesis is concerned with the Khovanov-Kuperberg algebras K^ε and to their categories of modules. They are analogous, in the $\mathfrak{sl}_3$ context, to the H^n algebras used by Khovanov to extend the $\mathfrak{sl}2_2$-homology (or Khovanov homology) to tangles. They appear as images of the 0-objects by a (0+1+1)-TQFT. They allow to define a $\mathfrak{sl}_3$-homology for tangles.

The categories of modules over these algebras are especially interesting: from their own construction they are deeply connected to bases in some representations of the quantum group $U_q(\mathfrak{sl}_3)$. It is hence natural to ask for a classification of the projective indecomposable modules over these algebras.

We study web modules which are projective K^ε-modules. It has been conjectured that these modules constitute a complete family of indecomposable projective K^ε-modules, but Khovanov and Kuperberg have exhibited a web module which decomposes as a direct sum. In this thesis we give two conditions on the indecomposability of web modules: a geometric sufficient condition and an algebraic necessary and sufficient condition.

The results are proven on the one hand, through a combinatorial analysis of webs which are plane bicubic graphs, and of a Laurent polynomial associated with each web called the Kuperberg bracket. And on the other hand, thanks to foams which plays the role of cobordisms for webs.

Keywords: Khovanov-Kuperberg algebras, $\mathfrak{sl}_3$-homology, Kuperberg bracket, Webs, Foams, TQFT, Colourings, Tangles.

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Introduction

Un peu d'Histoire[2]

Les nœuds ont une existence[3] préhistorique [TvdG96]. Leur utilité première est bien sûr mécanique et cette fonction est toujours d'actualité. L'administration inca les utilisait comme moyen de stocker de l'information (via les *quipus* [Hya90]). Ils sont à la base de toutes sortes d'ornements ou de décorations, notamment dans les civilisations chinoise, arabe et celte. Dans une certaine mesure, ils servent de signes religieux (le nœud sans fin ou *srivatsa* chez les boudhistes, les *tsitsits* chez les juifs). Notons aussi que Jacques Lacan illustre la relation Réel – Imaginaire – Symbolique (RIS) avec les anneaux borroméens [Lac11].

Malgré cette relative omniprésence des nœuds, leur apparition en mathématiques est assez tardive. En effet ce n'est qu'à la fin du XVIII$^{\text{ème}}$ siècle que Gauß les étudie pour la première fois (voir [Gau73a, Gau73b]). Dans les années 1870, James Clerk Maxwell, William Thompson (Lord Kelvin) et Peter Tait formulent une théorie qui postule que la nature d'un atome est déterminée par un nœud à l'intérieur de son noyau. Aussi s'attaquent-ils sérieusement au problème de classification [Tai98].

Le développement de la topologie algébrique par Poincaré a permis, d'une part de mieux poser le problème de classification, et d'autre part de profiter de tous les nouveaux outils que cette théorie propose : groupes fondamentaux, homologies, revêtements, etc... Il n'est pas raisonnable ici de donner un aperçu complet des travaux accomplis depuis lors. Citons tout de même Max Dehn et John Alexander qui s'intéressèrent particulièrement aux groupes fondamentaux des nœuds. Ce dernier donna son nom à un célèbre invariant polynomial. En 1926, Kurt Reidemeister [Rei27] donne un outil combinatoire : il montre que si deux diagrammes de nœuds représentent le même nœud alors on peut passer de l'un à l'autre par une suite finie de mouvements simples appelés *mouvements de Reidemeister*. Dans les années 50 et 60, Ralph Hartzler Fox a fait, entre autre, des liens importants entre les approches combinatoire et géométrique. Dans les années 70, John Conway définit les *enchevêtrements, i.e.* les nœuds à bords (voir figure 1) et le concept de *relations d'écheveaux*. Parallèlement il donne une nouvelle façon de noter les nœuds ce qui lui permet de construire de nouvelles méthodes pour le problème de classification.

Nous devons aussi mentionner que pendant toute cette période la théorie des tresses s'est développée et a fait des progrès de son côté avec des méthodes beaucoup plus algébriques (du fait de la structure de groupe). On peut par exemple citer la représentation de Burau. Même si le lien entre nœuds et tresses a été pressenti dès 1936 par Andreï Andreïevitch Markov, ce n'est qu'en 1974 que "son" théorème a été montré par Joan Birman [Bir74], permettant ainsi d'étudier les nœuds via les tresses.

En 1984 émerge une nouvelle branche de la théorie des nœuds : Vaughan Jones [Jon85]

[2] Nous conseillons vivement la lecture de [TvdG96].

[3] Il semble que les nœuds ne soient pas l'apanage des humains, voir par exemple [Her12].

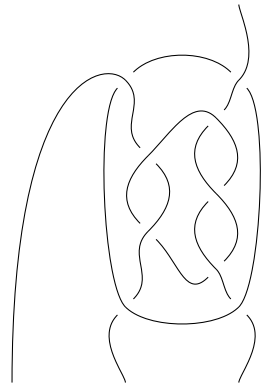


Figure 1 – Un enchevêtrement.

découvre un invariant polynomial grâce à des techniques totalement nouvelles. S'en est suivie une multitude de nouveaux invariants du même type que l'on a appelés *invariants quantiques*, citons le polynôme HOMFLYPT [FYH+85, PT88] et les invariants $\mathfrak{sl}_n$ [Tur88, MOY98]. Pour schématiser, l'idée est d'oublier la nature géométrique des nœuds, de se focaliser sur les diagrammes, et d'interpréter ces diagrammes comme des morphismes de modules sur une algèbre. Sous certaines conditions sur l'algèbre[4], les morphismes ainsi définis sont invariants par isotopie plane et par mouvements de Reidemeister. L'apparition de ces nouveaux invariants est une petite révolution. Jusque là l'étude des nœuds dépendait exclusivement (ou presque) des outils classiques de la topologie algébrique. Les invariants quantiques donnent un autre angle d'attaque, mais leur contenu géométrique est alors mal compris.

Les années 2000 donnent aux invariants quantiques une nouvelle dimension. Mikhail Khovanov [Kho00] explicite une *catégorification*[5] du polynôme de Jones. L'idée est de voir cet invariant comme le reflet de quelque chose de plus profond, *l'homologie de Khovanov* (ou homologie $\mathfrak{sl}_2$) : à un nœud on associe une suite de $\mathbb{Z}$-modules gradués dont la caractéristique d'Euler graduée est le polynôme de Jones de ce nœud. Toute la beauté et la puissance de cette construction réside dans le fait qu'elle est fonctorielle : un morphisme entre deux nœuds (*i.e.* un cobordisme) est envoyé sur une application entre les suites de $\mathbb{Z}$-modules gradués correspondant aux deux nœuds. La fonctorialité permet alors d'extraire des informations géométriques. Ainsi l'invariant de Rasmussen [Ras10], construit à partir d'une variante de l'homologie de Khovanov, donne une borne inférieure au genre lisse d'un nœud.

Après l'homologie de Khovanov, la question de la catégorification des autres invariants quantiques s'est posée. Des progrès substantiels ont été faits : les invariants $\mathfrak{sl}_n$ et le polynôme HOMFLYPT ont été catégorifiés [KR08a, KR08b] (par les homologies $\mathfrak{sl}_n$ et l'homologie HOMFLYPT). Dans cette thèse, nous nous intéressons à l'homologie $\mathfrak{sl}_3$, définie par Khovanov [Kho04] en 2004. Alors que les invariants quantiques polynomiaux ont des analogues pour les enchevêtrements, dans un premier temps leurs catégorifications n'étaient définis que pour les nœuds (sans bord). En 2002, Khovanov [Kho02] donne une version algébrique de l'homologie $\mathfrak{sl}_2$ pour les enchevêtrements. Par la suite Dror Bar-Natan [BN05] revisite l'homologie $\mathfrak{sl}_2$, et obtient lui aussi une version (moins algébrique) pour les

[4]On peut par exemple supposer que c'est une algèbre de Hopf quasi-triangulaire en ruban.

[5]Ici, par "catégorification" nous n'entendons pas d'énoncé mathématique précis, mais plutôt une démarche méta-mathématique. Pour une approche plus systématique de la catégorification, on pourra consulter le cours de Volodymyr Mazorchuk [Maz12].

enchevêtrements. En 2007, Scott Morrison et Ari Nieh [MN08] construisent une extension "à la Bar-Natan" de l'homologie $\mathfrak{sl}_3$ aux enchevêtrement.

Dans le chapitre 2 de cette thèse, nous donnons une version "à la Khovanov" de l'homologie $\mathfrak{sl}_3$ pour les enchevêtrements (voir aussi [MPT12] et [Rob12]). Les objets centraux de cette construction sont une famille d'algèbres appelées *algèbres de Khovanov-Kuperberg* que l'on note K^ε.

Il apparaît que ces algèbres ont des catégories de modules particulièrement intéressantes : du fait même de leur construction, elles sont profondément liées[6] à l'étude des bases de certaines représentations du groupe quantique $U_q(\mathfrak{sl}_3)$. Se pose alors le problème suivant :

Problème. *Trouver une collection complète de modules projectifs indécomposables sur l'algèbre K^ε.*

Le travail de cette thèse est sous-tendu par ce problème. Notons qu'un cap important a été franchi avec le calcul du groupe de Grothendieck scindé de K^ε par Mackaay, Pan et Tubbenhauer [MPT12].

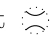
Les toiles et les mousses

L'homologie $\mathfrak{sl}_3$ a une définition très géométrique qui ressemble beaucoup à l'homologie de Khovanov. Étant donné un diagramme de nœud, on procède en 3 étapes :

1. Pour chaque croisement, on considère deux "résolutions" du croisement, et on définit un cobordisme entre les deux, on obtient ainsi un hypercube de résolutions du diagramme.

2. On transforme cet hypercube de résolutions en un hypercube de $\mathbb{Z}$-modules gradués en utilisant une TQFT adaptée.

3. On aplatit l'hypercube pour en faire un complexe de chaînes, et après certains décalages de degrés on prend l'homologie de ce complexe.

Pour l'homologie de Khovanov les résolutions de  sont  et , le cobordisme est une selle. Pour l'homologie $\mathfrak{sl}_3$, les résolutions et les cobordismes sont donnés par la figure 2.

Comme le montre la figure 2, les résolutions de diagrammes ne sont plus des courbes planes mais des graphes plans trivalents munis d'une orientation (ils sont en fait bipartites) avec d'éventuels cercles orientés (de tels objets sont appelés *toiles*). Les cobordismes ne sont plus de simples surfaces, mais des *mousses* qui sont des surfaces avec des cercles de singularité.

La catégorie source de la TQFT (étape 2) pour l'homologie $\mathfrak{sl}_3$ a pour objets les toiles et pour morphismes les mousses. Comme dans l'homologie de Khovanov, la TQFT est obtenue grâce à une construction universelle à la BHMV [BHMV95], en particulier ceci permet de voir l'image d'une toile par la TQFT comme un espace de combinaisons linéaires de mousses, modulo certaines relations. Pour l'homologie de Khovanov la TQFT était intimement liée à l'algèbre de Temperley-Lieb, qui spécifiait la dimension attendue de l'espace associé à une collection de cercles. Dans l'homologie $\mathfrak{sl}_3$ ce rôle de prescription est joué par un polynôme de Laurent associé à chaque toile appelé *crochet de Kuperberg.*

Ainsi l'étude des algèbres de Khovanov-Kuperberg peut prendre un tournant combinatoire (les toiles et le crochet de Kuperberg) ou géométrique (les mousses et leurs relations).

[6] Ceci avait déjà été remarqué avant que les algèbres de Khovanov-Kuperberg ne soient formellement définies : Khovanov et Kuperberg montrent que certaines bases ne sont pas duales canoniques [KK99].

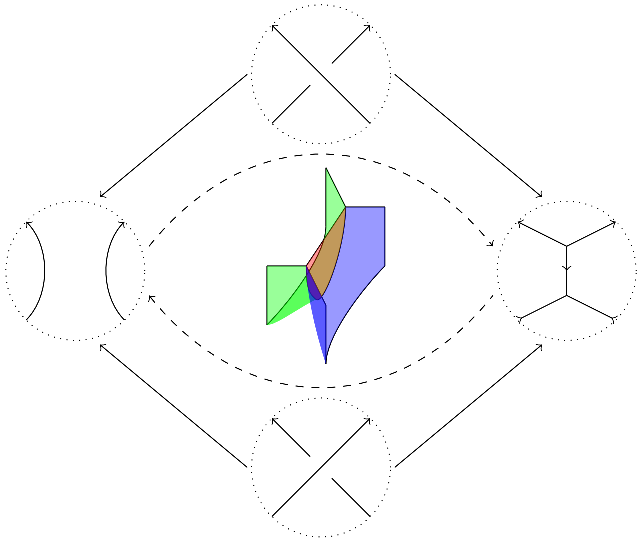


FIGURE 2 – Résolutions des croisements pour l'homologie $\mathfrak{sl}_3$.

Aux toiles sont naturellement associés des K^ε-modules projectifs appelés *modules de toiles*. Certaines toiles dites *non-elliptiques* sont irréductibles du point de vue de la combinatoire de Kuperberg. Il est alors naturel[7] d'espérer que les modules de toiles associés au toiles non-elliptiques donnent une solution au problème soulevé, mais avant même que la question ne soit posée en ces termes, Mikhail Khovanov et Greg Kuperberg [KK99] montrent que ce n'est pas le cas grâce à un contre-exemple (voir aussi [MN08] et la proposition 3.1.1) que nous reproduisons figure 3.

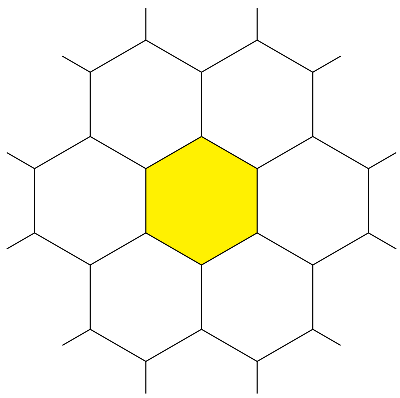


FIGURE 3 – Le contre-exemple de Khovanov et Kuperberg. La face nichée est jaune.

Il se trouve que la toile exhibée par Khovanov et Kuperberg est aussi l'exemple le plus simple de toile non-elliptique contenant une face nichée. Dans le chapitre 3 de cette thèse, nous montrons que ce n'est pas une coïncidence (voir le théorème 3.2.4) :

Théorème. *Si w est une toile non-elliptique n'ayant aucune face nichée alors le module de toile qui lui est associé est indécomposable.*

[7] En effet, dans le cas $\mathfrak{sl}_2$, les modules associés aux analogues des toiles non-elliptiques forment une collection complète de modules projectifs indécomposables [Kho02].

Ce résultat sur les modules de toiles est obtenu grâce à une analyse combinatoire fine des toiles et du crochet de Kuperberg qui indique, pour une toile donnée, la dimension graduée de l'espace des endomorphismes du module de toile associé. On utilise le fait que si l'espace des endomorphismes de degré 0 d'un module est de dimension 1 alors ce module est indécomposable. Dans le chapitre 4, nous montrons que s'agissant des modules de toiles la réciproque est vraie (voir le théorème 4.3.3) :

Théorème. *Si l'espace des endomorphismes de degré 0 d'un module de toile est de dimension strictement plus grande que 1, alors il est décomposable et contient un autre module de toile comme facteur direct.*

La preuve de ce résultat se fait en deux étapes : dans une première partie nous donnons une construction géométrique explicite de mousses correspondant à des idempotents non-triviaux. Dans une seconde, nous étudions la combinatoire des toiles pour déterminer sous quelles conditions une toile borde une telle mousse.

La construction permet d'exhiber de nombreux K^{ε}-modules projectifs indécomposables qui ne sont pas des modules de toiles. De plus, de par son caractère explicite, elle permet de formuler des réponses conjecturales au problème posé : dans tous les exemples que nous avons observés, les idempotents construits suffisent à décrire la décomposition des modules de toiles en sommes directe de modules projectifs indécomposables.

Le chapitre 5 est relativement indépendant : nous commençons par donner une description alternative du crochet de Kuperberg via des coloriages (cette construction est à rapprocher de [MOY98]). Ensuite nous établissons que tous les coloriages d'une toile sont proches dans un sens que nous définissons. Enfin nous appliquons ce résultat à l'étude de traces partielles dans la TQFT décrite au chapitre 2.

Introduction

A little history[8]

Knots exist[9] from prehistorical times. Their first use was for mechanical purpose, and this is still topical. The Inca administration used them as a mean to save data (with the talking knots or *quipus*). They are involved in quantities of decorations and ornaments, especially in Chinese, Arabic and Celtic civilisations. To some extent, knots are adopted as religious symbols: the endless knot or *srivasta* for Buddhists, the *tsitsits* for Jews can be quoted. Knots are not only pictural symbols but also illustrations of thought, for example, Jacques Lacan illustrated the Real – Imaginary – Symbolic (RIS) relation with the Borromean rings [Lac11].

Despite knots are nearly omnipresent, they appear quite late in mathematics. Indeed, Gauß studied them for the first time at the end of the XVIII$^{\text{th}}$ century (see [Gau73a, Gau73b]). In the 1870's, James Clerk Maxwell, William Thompson (Lord Kelvin) and Peter Tait formulated a physical theory where the nature of an atom is determined by a knot inside its kernel. Hence they started to tackle the classification problem (see [Tai98]).

Poincaré's development of algebraic topology allowed, on the one hand, to clarify what "classifying" means and on the other hand, to enjoy quantity of new tools proffered by this theory: fundamental groups, homologies, coverings, etc… It is utterly impossible to give here an exhaustive overview of all the accomplished work since then, however we would like to mention some of the most important steps.

Max Dehn and John Alexander were especially interested in the fundamental group of knots; a famous polynomial invariant was named after the latter. In 1926, Kurt Reidemeister [Rei27] gave a combinatorial tool to knot theory: he showed that two knot diagrams represent the same knot if and only if one can go from one to the other by a finite sequence of elementary moves called *Reidemeister moves*. In the 50's and the 60's, Ralph Hartzler Fox, among other things, pointed out some important links between the geometrical and the combinatorial approaches. In the 70's, John Conway defined the *tangles i.e.* knots with boundaries (see figure 4) and the concept of *skein relations*. Beside this, he gave a new way to encode knots. This permitted him to develop new methods for the classification problem.

We should as well mention that during this period braids theory developed and has carried off its own successes through more algebraic tools (this comes from the group structure of braids). One may quote, for example, the Burau representation. Even if the link between knots and braids has been foreseen already in 1936 by Andreï Andreïevitch Markov, it is only in 1974, that "his" theorem has be proven by Joan Birman [Bir74]. This allowed to study knots via braids.

In 1984, a new branch of knot theory sprang up: Vaughan Jones [Jon85], discover

[8] We warmly recommend [TvdG96].

[9] It seems that knots are not a humans' prerogative: see [Her12] for an (impressive) example.

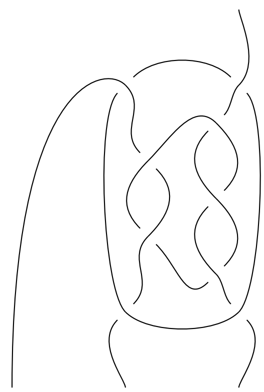


Figure 4: A tangle.

a new polynomial invariant by means of to completely new techniques. Then, a raft of invariants of the same kind followed and they were called *quantum invariants*, let us mention the HOMFLYPT polynomial [FYH+85, PT88] and the $\mathfrak{sl}_n$-invariants [Tur88, MOY98]. Roughly speaking, the idea is to forget the geometric nature of knots, and to focus on diagrams. Then one interprets these diagrams as morphisms of modules over a certain algebra[10]. The defined morphisms are shown to invariant under plane isotopy and under the Reidemeister moves. The emergence of these new invariants has been a small revolution. Until then, (almost) only the classical tools of algebraic topology were at knot theorists' disposal. The quantum invariant gave a new angle of attack but their geometrical meaning was at the time not clearly understood.

The 2000's gave to quantum invariants a new dimension. Mikhail Khovanov [Kho00], constructed explicitly a *categorification*[11] of the Jones polynomial. The idea is to see the Jones polynomial coming from something deeper, *the Khovanov homology* (or $\mathfrak{sl}_2$-homology): with a knot is associated a sequence of graded $\mathbb{Z}$-module whose Euler characteristic is the Jones polynomial of this knot. The beauty and the strength of this invariant comes from its functoriality: a morphism between two knots *i.e.* a cobordism, is associated with an application between the sequences of graded $\mathbb{Z}$-modules corresponding to the two knots. The functoriality permits to extract some geometrical information. Thus, Jacob Rasmussen[Ras10] constructed an invariant from a variant of the Khovanov homology which gives a lower bound to the slice genus of a knot.

After the Khovanov homology, arose the problem to find categorifications for the other quantum invariants. This quest has been rather successful: the $\mathfrak{sl}_n$ invariants and the HOMPLYPT polynomial have been categorified [KR08a, KR08b] (by the $\mathfrak{sl}_n$-homologies and the HOMFLYPT homology).

In this thesis, we are interested in the $\mathfrak{sl}_3$-homology, defined by Khovanov [Kho04] in 2004. Even though the polynomial quantum invariants extend immediately to tangles, their categorifications were first defined only for knots (without boundary). In 2002, Khovanov [Kho02] gave an algebraic version of the $\mathfrak{sl}_2$-homology for tangles. A little later, Dror Bar-Natan [BN05] revisited the $\mathfrak{sl}_2$ homology and obtained as well a version (less algebraic than Khovanov's one) for tangles. In 2007, Scott Morrison et Ari Nieh [MN08] constructed an extension "à la Bar-Natan" of the $\mathfrak{sl}_3$-homology for tangles.

In the second chapter of this thesis, we give a version "à la Khovanov" of the $\mathfrak{sl}_3$-

[10] One can, for example, work with a quasi-triangular ribbon Hopf algebra.

[11] By "categorification" we rather mean a meta-mathematical process than a precise mathematical statement. For a more systematical approach, we refer to the lecture of Volodymyr Mazorchuk [Maz12].

homology for tangles (see as well [MPT12] and [Rob12]). The central objetcs of this construction are a family of algebras called *Khovanov-Kuperberg algebras* and denoted by K^ε.

It appears that the categories of modules over these algebras are especially interesting: from their own construction, they are deeply connected[12] to bases in some representations of the quantum group $U_q(\mathfrak{sl}_3)$. Then the following question arises naturally:

Problem. *Exhibit a complete collection of projective indecomposable modules over the algebra K^ε.*

This problem underlies all the work of this thesis. Let us point out that significant improvements has been made by Mackaay Pan and Tubbenhauer [MPT12], since they managed to compute the split Grothendieck groups of the algebras K^ε.

Webs and foams

The $\mathfrak{sl}_3$-homology has a very geometrical definition and therefor looks a lot like the Khovanov homology. Being given a knot diagram, one proceeds in three steps:

1. For each crossing, one consider two "smoothtings" of the crossing, and one defines a cobordism between them. In the end, one obtains an hypercube of smoothings.
2. One turns this hypercube into an hypecube of graded $\mathbb{Z}$-module by means of an appropriate TQFT.
3. One flattens this hypercube, so that it becomes a chain complex, and after certain degree shifts, one compute the homology of this complex.

For the Khovanov homology the smoothings of are and , the cobordism is a saddle. For the $\mathfrak{sl}_3$ homology, the smoothings and the cobordisms are given by figure 5.

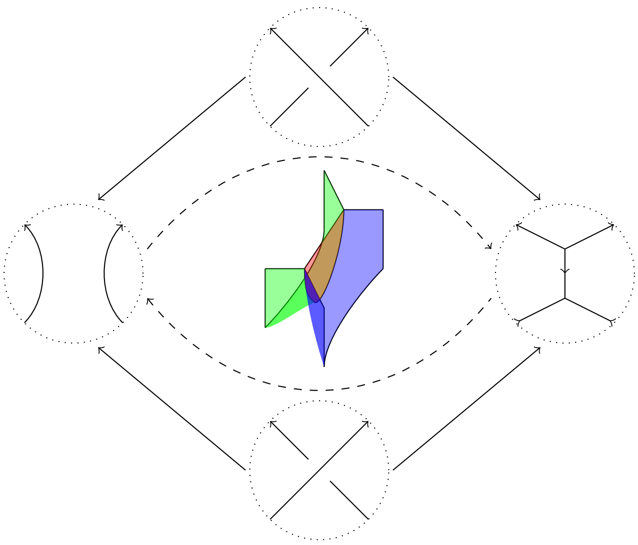


Figure 5: Smoothings of the crossings $\mathfrak{sl}_3$-homology.

As we can see on figure 5, the smoothings of the diagrams are no longer plane curves but oriented trivalent graphs (they are actually bipartite) with possible oriented circles.

[12] This has already been noticed before the Khovanov-Kuperberg algebras were properly defined: Khovanov and Kuperberg showed that some bases are not dual-canonical.

These objects are called *webs*. The cobordisms are not any more classical surfaces but *foams i.e.* surfaces with singularities along some circles.

The objects of the source category of the TQFT (step 2) for $\mathfrak{sl}_3$-homology are webs and its morphism are foams. Just as in Khovanov homology, the TQFT is obtained thanks to a universal construction à la BHMV [BHMV95], in particular this implies that the image of a web by the TQFT can be seen as space of linear combination of foams, modulo some relations. The Khovanov homology is closely related to the Temperley-Lieb algebra which specified the expected dimension of the space associated to a collection of circle. In the $\mathfrak{sl}_3$-homology this prescription role is played by a Laurent polynomial associated with every web and called *the Kuperberg bracket.*

Hence, to study the Khovanov-Kuperberg algebras, we dispose of a combinatorial approach (with the webs and the Kuperberg bracket) and a geometrical approach (with the foams and their relations).

With each web is naturally associated a projective K^ε-module named *web module.* Some webs, called *non-elliptic*, are irreducible with the combinatorial point of view arising from the Kuperberg bracket. Hence, it is natural[13] to hope that web modules associated with non-elliptic webs give a solution to the stated problem. However, even before the question was asked in those words, Mikhail Khovanov and Greg Kuperberg [KK99] have shown that this is not the case through a counter-example (see as well [MN08] and proposition 3.1.1). We reproduce this counter-example on figure 6.

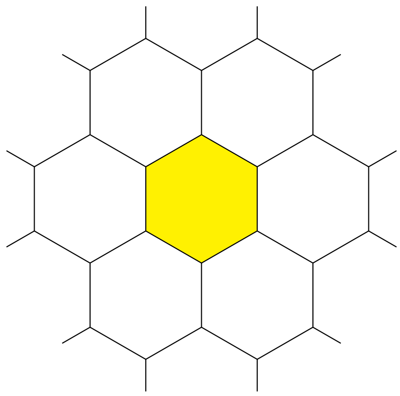


Figure 6: The counter-example of Khovanov and Kuperberg. The nested face is yellow.

It appears that the web exhibited by Khovanov and Kuperberg is as well the simplest example of a non-elliptic web with a nested face. In the third chapter of this thesis, we show that this is not a coincidence (see theorem 3.2.4):

Theorem. *If w is a non-elliptic web without any nested face, then the web module associated with w is indecomposable.*

This result on web modules is obtained by means of an accurate combinatorial analysis of webs and of the Kuperberg bracket which indicate for a given web the graded dimension of the space of endomorphisms of the associated web module. We use the fact that if the space of degree 0 endomorphism of a module is of dimension 1, then this module is indecomposable. In chapter 4, we prove that concerning web modules the reciprocal statement is true (see 4.3.3).

[13] Indeed, in the $\mathfrak{sl}_2$ case, the modules associated with the objects alike non-elliptic webs constitute a complete collection of projective indecomposable modules, see [Kho02].

Theorem. *If the space of endomorphisms with degree 0 of a web module has dimension strictly bigger than 1, then it is decomposable and it contains another web-module as a direct factor.*

The proof of this result is done in two steps: in a first part, we give an explicit geometrical construction of foams corresponding to non-trivial idempotents. In a second part, we study webs in a combinatorial way to detect the web-modules which admits such foams as endomorphisms.

The construction allows to exhibit many projective indecomposable K^{ε}-modules which are not web modules. Furthermore, due to its explicit nature, it permits formulate conjectural answers to the stated problem: in all the examples we computed, the constructed idempotents are enough to describe the decomposition of web modules as direct sum of projective indecomposable modules.

The chapter 5 is relatively independent from the rest: we begin by giving an alternative definition of the Kuperberg bracket through edge-colourings (this is to be compared with [MOY98]). Then we established that all the edge-colourings of webs are close in some sense. Finally we apply this result to compute partial traces in the TQFT described in chapter 2.

Chapter 1

Preliminaries

1.1 The Kuperberg bracket from representation theory

1.1.1 The Hopf-algebra $U_q(\mathfrak{sl}_3)$

The algebra $U_q(\mathfrak{sl}_3)$ is a deformation of the enveloping algebra of the Lie algebra $\mathfrak{sl}_3$ of null-trace 3×3-matrices of complex numbers. We give here a presentation of $U_q(\mathfrak{sl}_3)$ by generators and relations. It comes with a structure of Hopf-algebra.

Definition 1.1.1. The algebra $U_q(\mathfrak{sl}_3)$ is the associative $\mathbb{C}(q^{\frac{1}{2}})$-algebra[1] with unit generated by E_i, F_i, K_i and K_i^{-1} for $i = 1, 2$ and subjected to the relations for i and j in $\{1, 2\}$ and $i \neq j$:

$$\begin{aligned}
&K_iK_i^{-1} = K_i^{-1}K_i = 1, && K_1K_2 = K_2K_1,\\
&K_iE_i = q^2E_iK_i, && K_iF_i = q^{-2}F_iK_i,\\
&K_iE_j = -q^{-1}E_jK_i, && K_iF_j = qF_jK_i,\\
&(q^1 - q^{-1})(E_iF_i - F_iE_i) = K_i - K_i^{-1}, && E_iF_j = F_jE_i,\\
&E_i^2E_j - [2]E_iE_jE_i + E_jE_i^2 = 0, &&\\
&F_i^2F_j - [2]F_iF_jF_i + F_jF_i^2 = 0, &&
\end{aligned}$$

where $[n]$ stands for $\frac{q^n - q^{-n}}{q - q^{-1}}$. The co-unit $\eta : U_q(\mathfrak{sl}_3) \to \mathbb{C}(q^{\frac{1}{2}})$ the co-multiplication $\Delta : U_q(\mathfrak{sl}_3) \to U_q(\mathfrak{sl}_3) \otimes U_q(\mathfrak{sl}_3)$ and the antipode $S : U_q(\mathfrak{sl}_3) \to U_q(\mathfrak{sl}_3)$ are given by the following formulae for $i = 1, 2$:

$$\begin{aligned}
&\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, && \eta(K_i^{\pm 1}) = 1, && S(K_i^{\pm 1}) = K_i^{\mp 1},\\
&\Delta(E_i) = E_i \otimes 1 + K_i^{-1} \otimes E_i, && \eta(E_i) = 0, && S(E_i) = -K_iE_i,\\
&\Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i, && \eta(F_i) = 0, && S(F_i) = -F_iK_i^{-1}.
\end{aligned}$$

In the sequence we will consider two special $U_q(\mathfrak{sl}_3)$-modules: V^+ and V^-. Both are 3-dimensional and they are dual to each other. The module V^+ is the $U_q(\mathfrak{sl}_3)$-counterpart of the fundamental representation of $\mathfrak{sl}_3$. We detail in the next formulae the $U_q(\mathfrak{sl}_3)$-module structures of $V^+ = \left\langle e_{-1}^+, e_0^+, e_1^+ \right\rangle_{\mathbb{C}(q^{\frac{1}{2}})}$ and $V^- = \left\langle e_{-1}^-, e_0^-, e_1^- \right\rangle_{\mathbb{C}(q^{\frac{1}{2}})}$ on the generators (all

[1] The construction can be done on $\mathbb{C}(q)$ but $q^{\frac{1}{2}}$ allows to have more symmetry in the formulae.

the missing combinations are meant to be zero):

$$
\begin{array}{lll}
K_1 \cdot e^+_{-1} = e^+_{-1} & K_1 \cdot e^+_0 = -q^{-1} e^+_0 & K_1 \cdot e^+_1 = -q e_1 \\
K_2 \cdot e^+_{-1} = -q^{-1} e^+_{-1} & K_2 \cdot e^+_0 = -q e^+_0 & K_2 \cdot e^+_1 = e_1 \\
E_1 \cdot e^+_0 = e^+_1 & F_1 \cdot e^+_1 = e^+_0 & \\
E_2 \cdot e^+_{-1} = e^+_0 & F_1 \cdot e^+_0 = e^+_{-1} & \\
K_1 \cdot e^-_{-1} = -q^{-1} e^-_{-1} & K_1 \cdot e^-_0 = -q e^-_0 & K_1 \cdot e^-_1 = e^-_1 \\
K_2 \cdot e^-_{-1} = e^-_{-1} & K_2 \cdot e^-_0 = -q^{-1} e^-_0 & K_2 \cdot e^-_1 = -q e^-_1 \\
E_1 \cdot e^-_{-1} = e^-_0 & F_1 \cdot e^-_0 = e^-_{-1} & \\
E_2 \cdot e^-_0 = e^-_1 & F_1 \cdot e^-_1 = e^-_0 &
\end{array}
$$

If $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_l)$ is a finite sequence of signs, we denote by V^ε the $\mathbb{C}(q^{\frac{1}{2}})$-vector space $\bigotimes_{i=1}^{l} V^{\varepsilon_i}$ endowed with the structure of $U_q(\mathfrak{sl}_3)$-module provided by the co-multiplication and by the action of $U_q(\mathfrak{sl}_3)$ on V^+ and V^-. If ε is the empty sequence, then by convention V^ε is $\mathbb{C}(q^{\frac{1}{2}})$ with the structure of $U_q(\mathfrak{sl}_3)$-module given by the co-unit. We now define several maps between the V^ε's:

$$
\begin{array}{ll}
b^{+-} : \mathbb{C}(q^{\frac{1}{2}}) \to V^+ \otimes V^- & 1 \mapsto q e^+_{-1} \otimes e^-_1 + e^+_0 \otimes e^-_0 + q^{-1} e^+_1 \otimes e^-_{-1}, \\
b^{-+} : \mathbb{C}(q^{\frac{1}{2}}) \to V^- \otimes V^+ & 1 \mapsto q^{-1} e^-_{-1} \otimes e^+_1 + e^-_0 \otimes e^+_0 + q e^-_1 \otimes e^+_{-1}, \\
\sigma_{+-} : V^+ \otimes V^- \to \mathbb{C}(q^{\frac{1}{2}}) & e^+_{-1} \otimes e^-_1 \mapsto q, \quad e^+_0 \otimes e^-_0 \mapsto 1, \quad e^+_1 \otimes e^-_{-1} \mapsto q^{-1}, \\
\sigma_{-+} : V^- \otimes V^+ \to \mathbb{C}(q^{\frac{1}{2}}) & e^-_{-1} \otimes e^-_1 \mapsto q^{-1}, \quad e^+_0 \otimes e^-_0 \mapsto 1, \quad e^+_1 \otimes e^-_{-1} \mapsto q, \\
t^{+++} : \mathbb{C}(q^{\frac{1}{2}}) \to V^{(+,+,+)} & 1 \mapsto q^{\frac{-3}{2}} e^+_1 \otimes e^+_0 \otimes e^+_{-1} + q^{\frac{-1}{2}} e^+_0 \otimes e^+_1 \otimes e^+_{-1} \\
 & \quad + q^{\frac{-1}{2}} e^+_1 \otimes e^+_{-1} \otimes e^+_0 + q^{\frac{1}{2}} e^+_0 \otimes e^+_{-1} \otimes e^+_1 \\
 & \quad + q^{\frac{1}{2}} e^+_{-1} \otimes e^+_1 \otimes e^+_0 + q^{\frac{3}{2}} e^+_{-1} \otimes e^+_0 \otimes e^+_1, \\
t^{---} : \mathbb{C}(q^{\frac{1}{2}}) \to V^{(-,-,-)} & 1 \mapsto q^{\frac{-3}{2}} e^-_1 \otimes e^-_0 \otimes e^-_{-1} + q^{\frac{-1}{2}} e^-_0 \otimes e^-_1 \otimes e^-_{-1} \\
 & \quad + q^{\frac{-1}{2}} e^-_1 \otimes e^-_{-1} \otimes e^-_0 + q^{\frac{1}{2}} e^-_0 \otimes e^-_{-1} \otimes e^-_1 \\
 & \quad + q^{\frac{1}{2}} e^-_{-1} \otimes e^-_1 \otimes e^-_0 + q^{\frac{3}{2}} e^-_{-1} \otimes e^-_0 \otimes e^-_1.
\end{array}
$$

These maps are slightly different from the one defined in [KK99], but they are more symmetric in q and q^{-1}. They all respect the structures of $U_q(\mathfrak{sl}_3)$-modules.

Remark 1.1.2. The maps σ_{+-} and σ_{-+} fix an isomorphism between the dual of V^+ (resp. the dual of V^-) and V^- (resp. V^+). Under this isomorphisms the basis (e^+_{-1}, e^+_0, e^+_1) and $(q^{-1}e^-_1, e^-_0, qe^-_{-1})$ are dual to each other and the basis (e^-_{-1}, e^-_0, e^-_1) and $(qe^+_1, e^+_0, q^{-1}e^-_{-1})$ are dual to each other.

Following Reshetikhin-Turaev [RT90], we use a diagrammatic presentation of these maps (see figure 1.1, they should be read from bottom to top): a vertical strand represents the identity, stacking diagrams one onto another corresponds to composition, and drawing two diagrams side by side corresponds to taking the tensor product of two maps. The b's, σ's and t's allow us to define some other maps (e.g. $t^{++}_{-} \overset{\text{def}}{=} (\mathrm{id}_{V^{(+,+)}} \otimes \sigma_{+-}) \circ (t^{+++} \otimes \mathrm{id}_{V^-})$ coherent with the diagrammatic presentation (see figure 1.2).

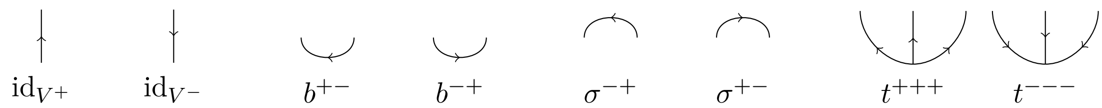


Figure 1.1: Diagrammatic description of b^{+-}, b^{-+}, σ_{-+}, σ_{+-}, t^{+++} and t^{---}.

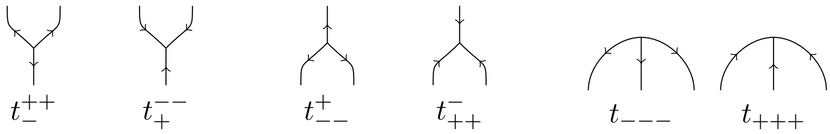


Figure 1.2: The maps t_{-}^{++}, t_{+}^{--}, t_{--}^{+}, t_{++}^{-}, t_{---} and t_{+++}.

Remark 1.1.3. It is a straightforward computation to check that the b's, the σ's and the $t's$ can be composed so that the diagrammatic representation makes sense. We mean that two isotopic diagrams represent the same map. One only need to check that the wave moves hold:

$$(\mathrm{id}_{V^+} \otimes \sigma_{-+}) \circ (b^{+-} \otimes \mathrm{id}_{V^+}) = \mathrm{id}_{V^+},$$

and similar equalities with signs and orders changed. It can be computed by hands.

1.1.2 Webs, web tangles and the Kuperberg bracket

In this subsection we develop the diagrammatic point of view and use the representation theory to define the Kuperberg bracket.

Definition 1.1.4 (Kuperberg, [Kup96])**.** A *closed web* is a cubic oriented graph (with possibly some vertexless loops) smoothly embedded in $\mathbb{R}^2$ such that every vertex is either a sink either a source.

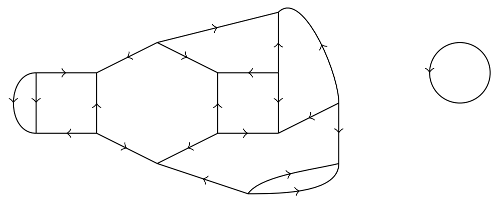


Figure 1.3: Example of a closed web.

Remark 1.1.5. By graph we don't mean simple graph, so that a web may have multi-edges. The orientation condition is equivalent to say that the graph is bipartite (by sinks and sources). The vertexless loops may be a strange notion from the graph theoretic point of view, to prevent this we could have introduced some meaningless 2-valent vertices[2].

Proposition 1.1.6. *Every closed web contains at least, a circle, a digon or a square.*

Proof. Let w be a closed web. One can suppose that the web w is connected for otherwise we consider could an innermost connected component. Suppose that it is not a circle. Then we use that the Euler characteristic of a connected plane graph is equal to 2:

$$\chi(w) = \#F(w) - \#E(w) + \#V(w) = 2,$$

[2]In this case the orientation condition is: around each vertex the flow module 3 is preserved.

where $F(w)$, $E(w)$ and $V(w)$ are the sets of faces, edges and vertices of w. Each vertex is the end of 3 edges, and each edge has 2 ends, so that $\#V(w) = \frac{2}{3}\#E(w)$. Now $\#F(w) = \sum_{i\in\mathbb{N}} F_i(w)$ where $F_i(w)$ denotes the number of faces of w with i sides. A web being bipartite, $F_i(w)$ is equal to 0 if i is odd. Each edges belongs to exactly two faces so that $\sum iF_i(w) = 2\#E(w)$. This gives:

$$\sum_i F_i(w) - \frac{i}{6}F_i(w) = 2.$$

And this shows that at F_2 or F_4 must be positive. □

Remark 1.1.7. Actually, we could have been more precise: if w is a connected closed web which is not a circle, then there are at least three[3] faces which have strictly less than 6 sides, furthermore, if w contains no digon, at least six[4] of its faces are squares.

Definition 1.1.8. Let $\varepsilon = \varepsilon^1, \varepsilon^2, \dots$ be a finite sequence of signs, *i.e.* of $+1$ and -1 The length of ε is denoted $l(\varepsilon)$. The sequence is said to be *admissible* if $\sum_{i=1}^{l(\varepsilon)} \varepsilon^i \equiv 0 \mod 3$.

Definition 1.1.9. A $(\varepsilon_0, \varepsilon_1)$-*web tangle* w is an intersection of a closed web w' with $\mathbb{R} \times [0,1]$ such that:

- there exists $\eta_0 \in]0,1]$ such that $w \cap \mathbb{R} \times [0,\eta_0] = \{1,2,\dots,l(\varepsilon_0)\} \times [0,\eta_0]$,
- there exists $\eta_1 \in [0,1[$ such that $w \cap \mathbb{R} \times [\eta_1,1] = \{1,2,\dots,l(\varepsilon_1)\} \times [\eta_1,1]$,
- the orientations of the edges of w, match $-\varepsilon_0$ and ε_1 (see figure 1.4 to have the conventions).

An ε-*web* is a $(\varepsilon, \emptyset)$-web tangle. If w is an ε-web, we define $\partial w \stackrel{\text{def}}{=} \varepsilon$. And we say that ε is *the boundary of* w. Note the flow modulo 3 is preserved everywhere on a web so that if ε is not admissible, then there exists no ε-web.

Figure 1.4: Examples of $(\varepsilon_0, \varepsilon_1)$-web tangles with $\varepsilon_0 = (+,+,+)$ and $\varepsilon_1 = (+,+,-,-)$.

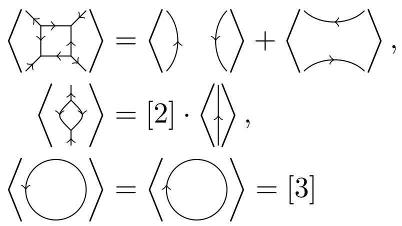
If w is a $(\varepsilon_0, \varepsilon_1)$-web tangle and w' is a $(\varepsilon_1, \varepsilon_2)$-web tangle we define *the composition* ww' to be the $(\varepsilon_0, \varepsilon_2)$-web tangle obtained by gluing w and w' along ε_1 and resizing.

Remark 1.1.10. The composition is not associative, this is associative only up to isotopy.

Notation 1.1.11. According to remark 1.1.3, one can associate a map of $U_q(\mathfrak{sl}_3)$-modules from V^{ε_0} to V^{ε_1} to any $(\varepsilon_0, \varepsilon_1)$-web tangle w. We denote by $\langle w \rangle$ this maps. We extend $\langle \cdot \rangle$ linearly to $\mathbb{C}(q^{\frac{1}{2}})$-linear combination of $(\varepsilon_0, \varepsilon_1)$-web tangles.

[3] Or two if one forbids the unbounded face.
[4] Or five if one forbids the unbounded face.

Proposition 1.1.12 (Kuperberg, [Kup96])**.** *We have the following equalities between maps of $U_q(\mathfrak{sl}_3)$-modules:*



The proof is a computation using the using the definitions of the maps σ's b's and t's given previously.

Definition 1.1.13. If w is a closed web, the $\langle w\rangle$ is can be seen as an element of $\mathbb{C}(q^{\frac{1}{2}})$. It can be checked (thanks to the relations of proposition 1.1.12) that this is actually always a symmetric Laurent polynomial in q. This is called the *Kuperberg bracket of w*.

As a consequence of 1.1.6 and 1.1.12, the Kuperberg bracket of a closed web can be computed completely via combinatorics (One can use [Lew11] to compute it on examples). The web depicted on figure 1.3 has its Kuperberg bracket equal to $2 \cdot [2]^5 \cdot [3]^2$.

Remark 1.1.14. The Kuperberg bracket of w evaluated in $q = 1$ gives the number of edge-3-colourings of w. One can actually define a degree for 3-edge-colourings in order to see the Kuperberg bracket as the graded number of 3-edge-colourings of w, see chapter 5 for details.

Definition 1.1.15. If w is a $(\varepsilon_0, \varepsilon_1)$-web tangle, we say that $\overline{w}$ is *the conjugate of w* if it is the $(\varepsilon_1, \varepsilon_0)$-web tangle obtained from w by taking the symmetric of w with respect to the line $\mathbb{R} \times \{\frac{1}{2}\}$ and then change all the orientations (see figure 1.5). It's clear that $\overline{\overline{w}} = w$.

Figure 1.5: These two web tangles are conjugate one to the other.

Notation 1.1.16. If ε is a sequence of signs we denote $\mathcal{B}_\varepsilon = \{b_i\}_{i\in I_\varepsilon}$ the base of V^ε given by the family

$$\left(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_{l(\varepsilon)}}\right)_{\{i_1,\dots,i_{l(\varepsilon)}\}\subset\{-1,0,1\}}.$$

If $b = e_{i_1} \otimes \cdots \otimes e_{i_{l(\varepsilon)}}$ is an element of $\mathcal{B}_\varepsilon$, we denote ${}^{\tau}b$ the element of $\mathcal{B}_\varepsilon$ equal to $e_{-i_1} \otimes \cdots \otimes e_{-i_{l(\varepsilon)}}$ We denote $\mathcal{B}^\star_\varepsilon = \{b^\star_i\}_{i\in I_\varepsilon}$ the dual base of $\mathcal{B}_\varepsilon$ (this is a base of $V^{-\varepsilon}$).

Proposition 1.1.17. *Let w be an $(\varepsilon_0, \varepsilon_1)$-web tangle then if:*

$$\langle w\rangle = \sum_{\substack{b_0^\star\in\mathcal{B}^\star_{\varepsilon_0}\\ b_1\in\mathcal{B}_{\varepsilon_1}}} \lambda_{b_1,b_0^\star} b_1 \otimes b_0^\star \quad \textit{and} \quad \langle\overline{w}\rangle = \sum_{\substack{b_1^\star\in\mathcal{B}^\star_{\varepsilon_1}\\ b_0\in\mathcal{B}_{\varepsilon_0}}} \mu_{b_0,b_1^\star} b_0 \otimes b_1^\star,$$

the λ's and μ's are Laurent polynomial in $q^{\frac{1}{2}}$ with integral non-negative coefficients which satisfy:

$$\lambda_{{}^{\tau}b_0,({}^{\tau}b_1)^\star}(q) = \lambda_{b_0,b_1^\star}(q^{-1}), \quad \mu_{{}^{\tau}b_1,({}^{\tau}b_0)^\star}(q) = \mu_{b_1,b_0^\star}(q^{-1}), \quad \textit{and} \quad \lambda_{b_0,b_1^\star} = \mu_{b_1,b_0^\star}.$$

Proof. These properties are stable by composition of webs and by performing tensor products of w with some vertical strands. Hence we just have to show it for the elementary webs depicted in figure 1.1. This is obviously true for the b's and the σ's. It remains to show it for t^{+++} and t^{---}. The point is then to compare t^{+++} (resp. t^{---}) and t_{+++} (resp. t_{---}). By definition we have:

$$t_{+++} = (\mathrm{id}_{V(+,+,+)} \otimes t^{---}) \circ (\mathrm{id}_{V(+,+)} \otimes \sigma_{+-} \otimes \mathrm{id}_{V(-,-)}) \circ (\mathrm{id}_{V^+} \otimes \sigma_{+-} \otimes \mathrm{id}_{V^-}) \circ \sigma_{+-}.$$

It is enough to compute t_{+++} on the element of $\mathcal{B}_{(+,+,+)}$ and we have:

$$\begin{aligned}
t_{+++}(e_1^+ \otimes e_0^+ \otimes e_{-1}^+) &= q^{-3/2}, & t_{+++}(e_0^+ \otimes e_{-1}^+ \otimes e_1^+) &= q^{-1/2},\\
t_{+++}(e_1^+ \otimes e_{-1}^+ \otimes e_0^+) &= q^{-1/2}, & t_{+++}(e_0^+ \otimes e_{-1}^+ \otimes e_1^+) &= q^{1/2},\\
t_{+++}(e_{-1}^+ \otimes e_1^+ \otimes e_0) &= q^{1/2}, & t_{+++}(e_{-1}^+ \otimes e_0^+ \otimes e_1^+) &= q^{3/2},
\end{aligned}$$

and the evaluations of t_{+++} on all other elements of $\mathcal{B}_{(+,+,+)}$ are equal to zero. So that we have:

$$\begin{aligned}
t_{+++} =& q^{-3/2}(e_1^+ \otimes e_0^+ \otimes e_{-1}^+)^\star + q^{-1/2}(e_0^+ \otimes e_{-1}^+ \otimes e_1^+)^\star\\
&+ q^{-1/2}(e_1^+ \otimes e_{-1}^+ \otimes e_0^+)^\star + q^{1/2}(e_0^+ \otimes e_{-1}^+ \otimes e_1^+)^\star\\
&+ q^{1/2}(e_{-1}^+ \otimes e_1^+ \otimes e_0^+)^\star + q^{3/2}(e_{-1}^+ \otimes e_0^+ \otimes e_1^+)^\star.
\end{aligned}$$

Comparing this formula to the definition of t^{+++} given page 2, gives the result for t^{+++}. It remains to check the result for t^{---} for which the computations are similar. □

Suppose that w is an ε-web, then we can write $\langle w \rangle = \sum_{b\in\mathcal{B}_\varepsilon} \lambda_b b^\star$, but with these notations we have $\langle \overline{w} \rangle = \sum_{b\in\mathcal{B}_\varepsilon} \lambda_b b$, so that we have:

$$\langle \overline{w} w \rangle = \sum_{b\in\mathcal{B}_\varepsilon} \lambda_b^2.$$

Notation 1.1.18. We denote by $\mathbb{Z}[q,q^{-1}]_s$ (resp. $\mathbb{N}[q,q^{-1}]_s$) the space of symmetric Laurent polynomial (resp. , the space of symmetric Laurent polynomial with non-negative coefficient). The degree $\deg(P)$ of a symmetric Laurent polynomial is defined to be the degree of its polynomial part.

Proposition 1.1.19. *Let $(w_i)_{i\in I}$ a finite collection of ε-web and $(P_i)_{i\in I}$ a collection of non-zero Laurent polynomials in $\mathbb{N}[q,q^{-1}]_s$. Let $W = \sum_{i\in I} P_i w_i$ and $\overline{W} = \sum_{i\in I} P_i \overline{w_i}$. Then $\left\langle \overline{W} W \right\rangle$ belongs to $\mathbb{N}[q,q^{-1}]_s$, its evaluation on any non-zero real number is non-negative, and if it has degree n then there exists $i_0 \in I$ such that $P_{i_0}^2 \langle \overline{w_{i_0}} w_{i_0} \rangle = \langle P_{i_0} \overline{w_{i_0}} P_{i_0} w_{i_0} \rangle$ has degree n.*

Proof. The fact that $\left\langle \overline{W} W \right\rangle = \sum P_i P_j \langle \overline{w_i} w_j \rangle$ is a symmetric Laurent polynomial with non-negative coefficients follows from the fact that for any closed web w, $\langle w \rangle$ is a symmetric

Laurent polynomial with non-negative coefficients. For each web w_i we set: $\langle w_i \rangle = \sum_{b\in\mathcal{B}_\varepsilon} \lambda_{b,i} b^\star$, we then have:

$$\left\langle \overline{W}W \right\rangle = \sum_{b\in\mathcal{B}_\varepsilon i,j\in I} P_i\lambda_{b,i}P_j\lambda_{b,j}.$$

Let i_0 be an element of I and b_0 and element of $\mathcal{B}$ such that the degree of $P_{i_0}\lambda_{b_0,i_0}$ is maximal (this Laurent polynomial need not to be symmetric, we consider the degree of the polynomial part of it), as every coefficient is positive, we have:

$$\deg \left\langle \overline{W}W \right\rangle = \deg(P_{i_0}\lambda_{b_0,i_0})^2 = \deg P_{i_0}^2 \left\langle \overline{w_{i_0}} w_{i_0} \right\rangle.$$

□

Definition 1.1.20. Let ε be an admissible sequence of signs. We denote S^ε the $\mathbb{Z}[q,q^{-1}]$-module generated by ε-webs modulo isotopy and quotiented by the relations of 1.1.12.

Definition 1.1.21. An $(\varepsilon_0, \varepsilon_1)$-web tangle or an ε-web is said to be *non-elliptic* if it contains no vertex-less loop, no digon and no square.

Theorem 1.1.22 (Kuperberg,[Kup96])**.** *Let ε be an admissible sequences of signs, then the set $(\langle w \rangle)_{w\in NE(\varepsilon)}$ is a base of $\hom_{U_q(\mathfrak{sl}_3)}(V^\varepsilon, \mathbb{C}(q^{\frac{1}{2}}))$ where $NE(\varepsilon)$ is a set of representatives of isotopy classes of non-elliptic ε-webs.*

This theorem implies that for a fixed ε they are finitely many isotopy classes of non-elliptic ε-webs and that the following proposition holds:

Proposition 1.1.23. *The $\mathbb{Z}[q,q^{-1}]$-module S^ε is free and the non-elliptic ε-webs form a basis called the* Kuperberg basis*. Furthermore if an ε-web $w = \sum P_i w_i$ with w_i non-elliptic webs, then the P_i's are unique have only non-negative coefficient and are symmetric in q and q^{-1}.*

1.2 The $\mathfrak{sl}_3$-TQFT

We fix R to be a commutative ring with unit, the orignal construction is done over $\mathbb{Z}$ and in chapter 4 we will need to work over $\mathbb{Q}$ (actually over any a field of characteristic 0).

In this section we recall a construction of [Kho04] (see [MV07] or [MN08] for alternative descriptions). We first describe the category Foam, which is a generalisation of the category Cob_{1+1} of cobordisms where instead of 1-manifolds we have webs. Then we will explain the constuction of the TQFT-functor of Khovanov [Kho04] from Foam to the category of graded R-module.

Definition 1.2.1. A *pre-foam* is a smooth oriented compact surface Σ (its connected components are called *facets*) together with the following data:

- A partition of the connected components of the boundary into cyclically ordered 3-sets and for each 3-set (C_1, C_2, C_3), three orientation preserving diffeomorphisms $\phi_1 : C_2 \to C_3$, $\phi_2 : C_3 \to C_1$ and $\phi_3 : C_1 \to C_2$ such that $\phi_3 \circ \phi_2 \circ \phi_1 = \mathrm{id}_{C_2}$.
- A function from the set of facets to the set of non-negative integers (this gives the number of *dots* on each facet).

The *CW-complex associated with a pre-foam* is the 2-dimensional CW-complex Σ quotiented by the diffeomorphisms so that the three circles of one 3-set are identified and become one circle called a *singular circle*. The *degree* of a pre-foam f is equal to $-2\chi(\Sigma')$ where χ is the Euler characteristic, Σ' is the CW-complex associated with f with the dots punctured out (*i.e.* a dot increases the degree by 2).

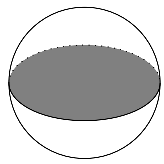


Figure 1.6: Example of a prefoam: the dotless theta pre-foam.

Remark 1.2.2. The CW-complex has two local models depending on whether we are on a singular circle or not. If a point x is not on a singular circle, then it has a neighborhood diffeomorphic to a 2-dimensional disk, else it has a neighborhood diffeomorphic to a Y shape times an interval (see figure 1.7).

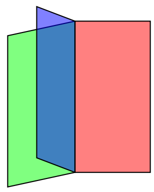


Figure 1.7: Singularities of a pre-foam.

Definition 1.2.3. A *closed foam* is the image of an embedding of the CW-complex associated with a pre-foam such that the cyclic orders of the pre-foam are compatible with the left-hand rule in $\mathbb{R}^3$ with respect to the orientations of the singular circles[5] (see figure 1.8). The *degree* of a closed foam is the degree of the underlying pre-foam.

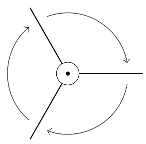


Figure 1.8: Orientation condition of closed foams. This figure represents a cutted view around a singular circle. The singular circle is locally oriented from bottom to top (orientation represented by $\odot$), while the thin arrows indicate how the 3 facets are cyclically ordered.

Definition 1.2.4. If w_b and w_t are closed web, a (w_b, w_t)*-foam* f is the intersection of a foam f' with $\mathbb{R} \times [0,1] \times [0,1]$ such that

- there exists $\eta_b \in]0,1]$ such that $f \cap \mathbb{R} \times [0,1] \times [0,\eta_b] = w_b \times [0,\eta_b]$,

[5] We mean here that if, next to a singular circle, with the forefinger of the left hand we go from face 1 to face 2 to face 3 the thumb points to indicate the orientation of the singular circle (induced by orientations of facets). This is not quite canonical, physicists use in general the right-hand rule, however this is the convention used in [Kho04].

- there exists $\eta_t \in [0,1[$ such that $f \cap \mathbb{R} \times [0,1] \times [\eta_t, 1] = w_t \times [\eta_t, 1]$,

with compatibility of orientations of the facets of f with the orientation of w_t and the reversed orientation of w_b. The *degree* of a (w_b, w_t)-foam f is equal to $\chi(w_b)+\chi(w_t)-2\chi(\Sigma)$ where Σ is the underlying CW-complex associated with f with the dots punctured out.

Definition 1.2.5. The category Foam is the category whose objects are closed webs and whose morphisms between w_b and w_t are finite $R[q^{-1}, q]$-linear combinations of isotopy classes of (w_b, w_t)-foam

The construction of the TQFT relies on a strategy developed in [BHMV95] called *universal construction*. The idea is to construct a numerical invariant for closed cobordisms (here closed foams) and thanks to this numerical[6] invariant to define a TQFT. The functoriality of the construction will be straightforward while the finite dimensional property will be the important point to check.

Collection of circles and surfaces are special cases of webs and foams, so that the TQFT arising from the construction will be in particular a "classical" TQFT. We begin by settling this up.

Definition 1.2.6. We denote by $\mathcal{A}$ the Frobenius algebra $R[X]/(X^3)$ with trace τ given by:

$$\tau(X^2) = -1, \quad \tau(X) = 0, \quad \tau(1) = 0.$$

We equip $\mathcal{A}$ with a graduation by setting $\deg(1) = -2$, $\deg(X) = 0$ and $\deg(X^2) = 2$. With these settings, the multiplication has degree 2 and the trace has degree -2. The co-multiplication is determined by the multiplication and the trace and we have:

$$\begin{aligned}
\Delta(1) &= -1 \otimes X^2 - X \otimes X - X^2 \otimes 1 \\
\Delta(X) &= -X \otimes X^2 - X^2 \otimes X \\
\Delta(X^2) &= -X^2 \otimes X^2
\end{aligned}$$

This Frobenius algebra gives us a (1+1)-TQFT (this is well-known, see [Koc04] or [Kad99] for details), we denote it by $\mathcal{F}$: the circle is sent to $\mathcal{A}$, a cup to the unity, a cap to the trace, and a pair of pants either to multiplication or co-multiplication. A dot on a surface represents multiplication by X so that $\mathcal{F}$ extends to the category of oriented dotted (1+1)-cobordisms. We then have a surgery formula given by figure 1.9. In particular

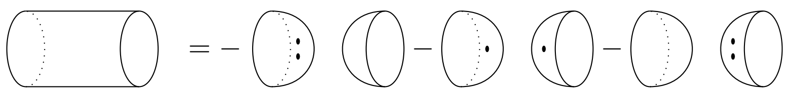
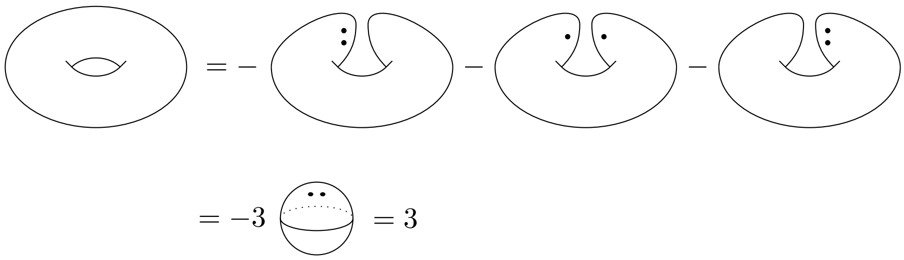


Figure 1.9: The surgery formula for the TQFT $\mathcal{F}$.

this TQFT gives rise to a numerical invariant for closed surfaces (still denoted by $\mathcal{F}$). To compute this invariant one just need the surgery relation and the fact that a sphere evaluates to 0 unless is carries 2 dots and in this circumstance its evaluation is equal to -1.

[6]By numerical we will always mean R-valued, where R is the commutative ring choose at the beginning of this section.

For example the value affected to a dotless torus is equal to 3:



Proposition 1.2.7. *There exists a unique extension (still denoted by $\mathcal{F}$) of the numerical invariant $\mathcal{F}$ to pre-foams satisfying the following conditions:*

- *$\mathcal{F}$ is multiplicative with respect to the disjoint union of pre-foams.*
- *$\mathcal{F}$ satisfies the surgery formula for pre-foams, meaning that if f, f_1, f_2 and f_3 are the same pre-foams except in a small ball where they looks like the terms in the formula of the figure 1.9, then we have $\mathcal{F}(f) = -\mathcal{F}(f_1) - \mathcal{F}(f_2) - \mathcal{F}(f_3)$.*
- *The values of $\mathcal{F}$ of dotted theta pre-foams is given by figure 1.10.*

Figure 1.10: The evaluations of dotted theta pre-foams, the cyclic order on facets being top < middle < bottom (this fits the convention of figure 1.8). Theta pre-foams with other dots configurations are sent to 0.

Proof. It's clear that for any pre-foam the relations given are enough to compute the value: with the surgery formula, one can separate the pre-foam into a collection of theta pre-foams and of closed surfaces, and for such pre-foam the relation gives directly the invariant, this proves the uniqueness of such an invariant.

To prove the existence, one should check that these relations are consistent. The consistency on dotted surfaces comes from the fact that $\mathcal{F}$ is a functor. The only way to evaluate $\mathcal{F}$ on pre-foams with this rules, is to separate this foams into (dotted) theta pre-foams and surfaces, one only has to check that the evaluation does not change when one first performs a surgery on a theta pre-foam and then evaluates, or just evaluate the theta pre-foam. This is can be checked by an easy computation. □

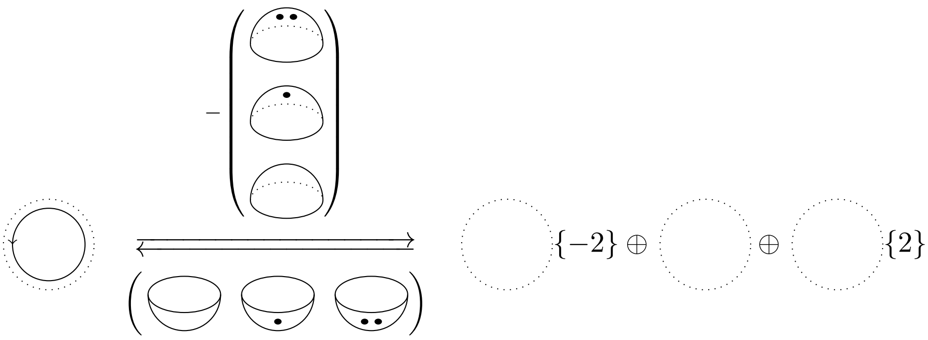
Remark 1.2.8. Note that the surgery is a homogeneous relation and that all non-trivial evaluation of $\mathcal{F}$ on dotted theta pre-foam concerns degree 0 pre-foams. This is true as well for the evaluation on dotted surfaces, so that if f is a pre-foam such that $\mathcal{F}(f) \neq 0$, then f has degree 0.

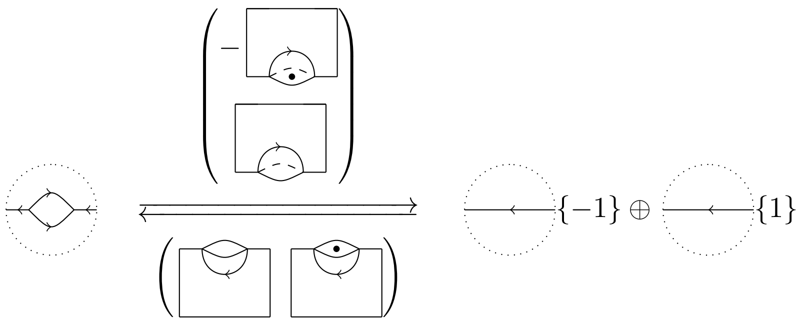
Definition 1.2.9 (Universal construction à la BHMV)**.** Let w be a web, we consider the graded R-module $\mathcal{G}(w)$ freely generated by any $(\emptyset, w)$-foams (grading comes from the grading of foams). This is endowed with a bi-linear form $< \cdot, \cdot >$: if (f, g) is a pair of $(\emptyset, w)$-foams, then we define $< f, g >$ to be $\mathcal{F}(\overline{g}f)$ where $\mathcal{F}$ is the numerical invariant and $\overline{g}f$ is the foam obtained by gluing f and the mirror image of g along w. We defined $\mathcal{F}(w)$ to be $\mathcal{G}(w)/\operatorname{Ker} < \cdot, \cdot >$.

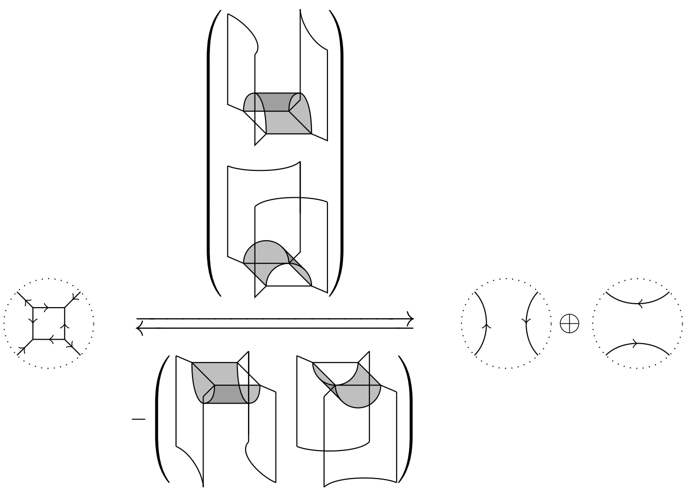
Note that $\mathcal{G}(w)$ is still graded because of remark 1.2.8. Note as well that this construction gives the value of the functor $\mathcal{F}$ on (w_1, w_2)-foams: a (w_1, w_2)-foam f can be naturally seen as a linear map from $\mathcal{G}(w_1)$ to $\mathcal{G}(w_1)$ with degree $\deg f$, and this map respects the quotient by Ker $< \cdot, \cdot >$.

Theorem 1.2.10 (Khovanov, [Kho04]). *Let w be a web, then $\mathcal{F}(w)$ is a free graded R-module of graded dimension equal to $\langle w \rangle$.*

The statement is actually more precise: Khovanov gives explicit isomorphism corresponding to the relation of proposition 1.1.12: he shows that the following 3 pairs of morphisms are mutually inverse isomorphisms (the brackets stand for degree shifts):







To show this he first proves that some locale relation holds. We give some of them on figure 1.11 (see page 13).

These three isomorphisms together with proposition 1.1.6 gives a way to compute an homogeneous base of the space $\mathcal{F}(w)$ for any closed web w (note that one has to make choices so that the computed base is not unique).

These three isomorphisms together with proposition 1.1.6 and the fact that the surgery, the evaluations of dotted sphere and the dotted theta foams are enough to compute the value of any closed foams shows the following proposition:

Proposition 1.2.11. *We consider the set* ***FR*** *of local relations which consists of:*

- *the surgery relation,*
- *the evaluations of the dotted spheres and of the dotted theta-foams,*
- *the square relations and the digon relations (see figure 1.11).*

We call them the foam relations *or relations* ***FR****, then for any closed web w $\mathcal{F}(w)$ is isomorphic to $\mathcal{G}(w)$ modded out by* ***FR****.*

1.3 The $\mathfrak{sl}_3$-homology for links

Using this TQFT functor, Khovanov [Kho04] constructs an homological invariant for oriented links which categorifies the $\mathfrak{sl}_3$-polynomial: we explain here the construction. We do not repeat the proof of invariance under Reidemeister moves.

1.3.1 Smoothings

Definition 1.3.1. Let D be an oriented link diagram. *A smoothing function for D* is a function ϕ from the set of crossings of D to $\{0, 1\}$. Let c be a crossing of D and ϕ a smoothing function for D such that $\phi(c) = 0$, we denote by ϕ_c the smoothing function equal to ϕ everywhere but on c where we have $\phi_c(c) = 1$. The *size of ϕ* is the number of times it takes the value 1. We denote it by $|\phi|$.

Definition 1.3.2. Let D be an oriented link diagram and ϕ a smoothing function for D. Then we define *the ϕ-smoothing of D* to be the closed web obtain from D by replacing each crossing c by its $\phi(c)$-smoothing (see figure 1.12 for definitions of 0-smoothing and 1-smoothing). We denote it by D_ϕ.

1.3.2 The chain complex

Definition 1.3.3. Let I be a finite set and $\mathcal{C}$ an additive category, then a *naive I-hypercube in $\mathcal{C}$* is a family $(X_\phi)_{\phi\in\{0,1\}^I}$ of object of $\mathcal{C}$ together with a family of maps $d^i_\phi : X_\phi \to X_{\phi_i}$ where ϕ is in $\{0, 1\}^I$ i is in I such that $\phi(i) = 0$, and ϕ_i is the same application except that $\phi_i(i) = 1$, and the compositions of any two pairs of maps are compatible, we mean, than whenever it makes sense:

$$d^k_{\phi_i} \circ d^i_\phi = d^i_{\phi_k} \circ d^k_\phi. \tag{1.1}$$

As before, we denote $\sum_{i\in I} \phi(i)$ by $|\phi|$.

Definition 1.3.4. Given I a finite set, $H = (X_*, d_*)$ a naive I-hypercube in an additive category and $\prec$ a total order on I one can form *the complex $C_*(H, \prec)$* by setting for all n in $[0, \#I]$:

$$C_n = \bigoplus_{\phi \text{ such that } |\phi|=n} X_\phi$$

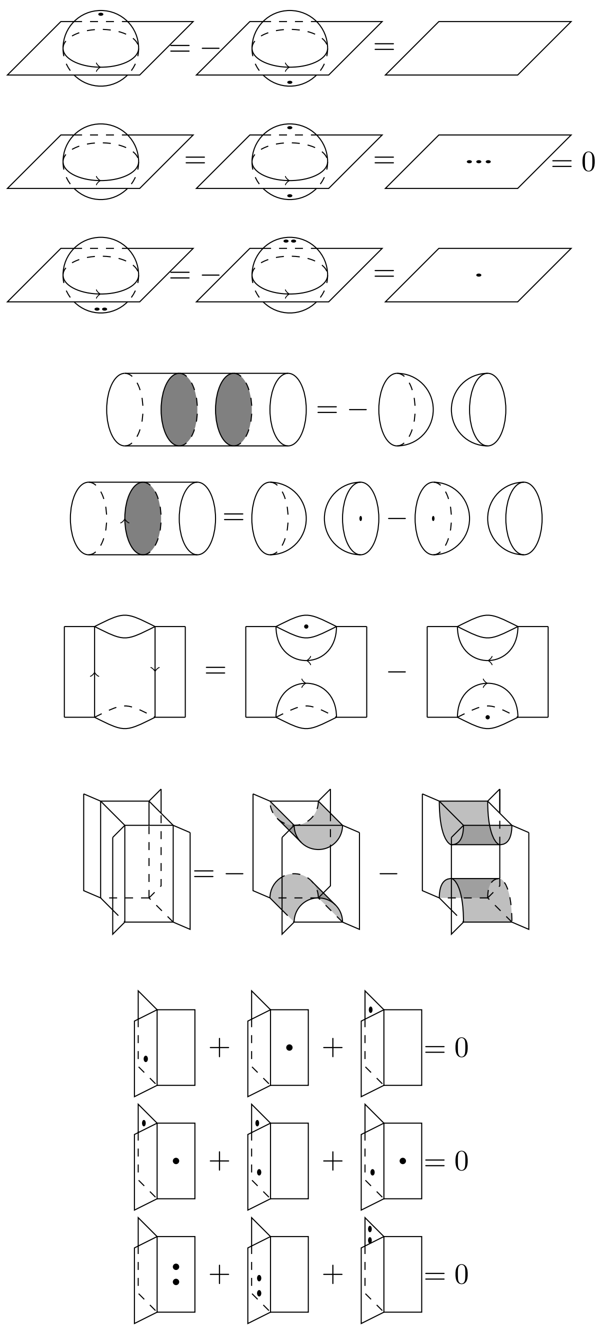


Figure 1.11: The first 3 lines are called bubbles relations, the 2 next are called bamboo relations, the one after digon relation, then we have the square relation and the 3 last ones are the dots migration relations.

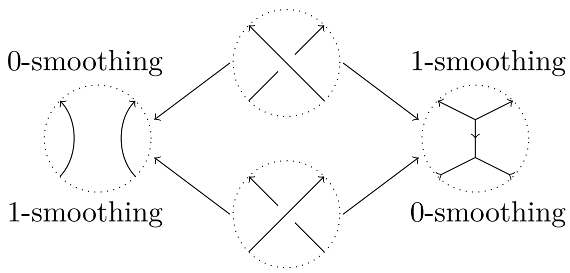


Figure 1.12: Definition of the 0-smoothing and the 1-smoothing.

Remark 1.3.5. This definition is designed to change the commutation formula for I-hypercubes to an anti-commutation relation so that it's straightforward that $C(H, \prec)$ is a complex. For $\prec_1$ and $\prec_2$ to give order, there exits a canonical isomorphism from $C(H, \prec_1)$ to $C(H, \prec_2)$. For I a finite set, one can choose any order $\prec$ and construct $C(H, \prec)$, this is why, in the sequel, we will forget to mention the order and just write $C(H)$, considering one of all possible order on I. For further discussion on signs convention we refer to Deligne[MASD73, Exposé 17]. Strictly speaking this is a co-chain complex, therefor one should speak about $\mathfrak{sl}_3$-cohomology, however we follow the vocabulary from the community.

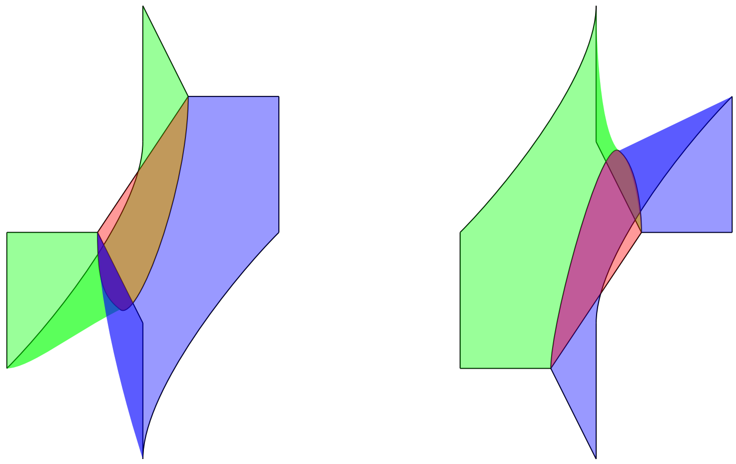


Figure 1.13: Singular Saddles (called as well zip and unzip). These are the non-trivial parts of the foams inducing the differentials in the complex for positive crossings on the left, and for the negative crossings on the right (the cobordisms are read from bottom to top).

Definition 1.3.6. Let D be link diagram. Let I be the set of crossings of D. We define $H(D)$, *the hypercube of smoothings of* T, to be the I-hypercube where H_ϕ is the image by the functor $\mathcal{F}$ of the smoothing ϕ of D with a degree shift equal to minus the length of ϕ (note that with this degree shift the differential are homogeneous). For ϕ a smoothing function and c a crossing such that $\phi(c) = 0$, the differential d_ϕ^c is the images by $\mathcal{F}$ of the foam which is everywhere identity but next to the crossing c where it is given by figure 1.13.

It's clear that the differential satisfy the compatibility relation because the foams representing differentials have their non-trivial pieces on different places, so that the compositions in any order are homotopic, and hence the induced maps are equal. The *complex* $C(D)$ *of smoothing of* D is the complex $C(H(D))\{3n_- - 2n_+\}[-n_-]$ where $[\cdot]$

denotes an homological degree shift, $\{\cdot\}$ the degree shift and n_+ (resp. n_-) the number of positive (resp. negative) crossings of the diagram D.

Theorem 1.3.7 (Khovanov [Kho04])**.** *If two link diagrams represent the same link, then, their complexes of smoothing are homotopic.*

To prove this theorem, Khovanov exhibits some homotopy equivalences between complexes of smoothings of diagrams related by Reidemeister moves.

Chapter 2

The $\mathfrak{sl}_3$-homology for tangles

The aim of this chapter is to define the $\mathfrak{sl}_3$-homology for tangles. This is a functor-valued invariant which extend to tangles the $\mathfrak{sl}_3$-homology for links defined in [Kho04]. In this purpose, we mimic the strategy of [Kho02] and define a (0+1+1)-TQFT. The central objects of this construction are the algebroids $\mathbb{K}^\varepsilon$ (or the algebras K^ε). They both come from some algebroids big $\widetilde{K^\varepsilon}$. Even if for simplicity we will just work with the algebra versions of these objects, we recall some facts on algebroids because they are the natural framework to work with. Then we detail the construction of the (0+1+1)-TQFT and of the $\mathfrak{sl}_3$-homology for tangles. A short version of this chapter can be found in the first part of [Rob12]. We emphasise that the algebras $\mathbb{K}^\varepsilon$ were defined independently by Mackaay, Pan and Tubbenhauer in [MPT12].

2.1 Reminder on algebroids

The structure of algebroid is a generalisation of the structure of algebra. If one sees an algebra as the set of endomorphisms of an object in an appropriate category (the product being transposed to the composition), it is natural to consider the whole category instead of just one object. This leads to the notions of algebroids. We give two different approaches to algebroids the first one is the classical one, the second is less elegant but somewhat easier to manipulate. In this section k will be a commutative ring.

2.1.1 The genuine algebroids from category theory

All the material here comes from [Mit85]. For the set theoretical issues we refer to [KS06], [ML98] and [GV72, Exposé 1, section 0].

Definition 2.1.1. A category $\mathcal{C}$ is *small* if $\mathrm{Ob}(\mathcal{C})$ and $\hom(\mathcal{C})$ are sets. A *locally small category* $\mathcal{D}$ is a category such that for every pair of objects (a, b) of $\mathcal{D}$, $\hom(a, b)$ is a set. A category which is not small is *large*.

Definition 2.1.2. A *k-category* is a locally small category such that each of its homset is given a k-module structure and such that the composition of morphisms is a k-bilinear map. A *k-functor* between two k-categories $\mathcal{C}_1$ and $\mathcal{C}_2$ is a functor F from $\mathcal{C}_1$ to $\mathcal{C}_2$ such that for any objects a and b the map induced by F from $\hom_{\mathcal{C}_1}(a, b)$ to $\hom_{\mathcal{C}_2}(Fa, Fb)$ is k-linear.

The composition of two k-functors is a k-functor and the identity functor is a k-functor so that there is a natural notion of (large) category of k-categories. One can also define a

k-linear category to be a category enriched over the category of k-modules, in this context a k-functor is an enriched functor.

Example 2.1.3. A $\mathbb{Z}$-category is an additive category, and a $\mathbb{Z}$-functor is an additive functor. If $\mathcal{A}$ and $\mathcal{A}'$ are two k-categories then the *tensor product of* $\mathcal{A}$ *and* $\mathcal{A}'$ is the k-category $\mathcal{A} \otimes \mathcal{A}'$ defined by:

$$\begin{aligned} \mathrm{Ob}(\mathcal{A} \otimes \mathcal{A}') &= \mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{A}'), \\ \hom_{\mathcal{A}\otimes\mathcal{A}'}\left((a,a'),(b,b')\right) &= \hom_{\mathcal{A}}(a,b) \otimes \hom_{\mathcal{A}'}(a',b'), \end{aligned}$$

where the tensor product is over k (all unadorned tensor products mean tensor products over k).

Definition 2.1.4. A *k-algebroid* is a small k-category. A *k-algebra* is a k-algebroid with one object.

Example 2.1.5. Let n be a positive integer, the algebroid k_n is defined as follows: the objects are $1, \dots, n$, and $\hom(n_1, n_2) = k$ for all n_1 and n_2 in $[1, n]$. The composition is given by the multiplication of k. As we shall see $\mathcal{M}_n(k)$ is the regularized algebra of k_n. The ring k is considered as a k-algebroid (actually as an algebra) via k_1. Note that if $\mathcal{A}$ is a k-algebroid, the opposite category $\mathcal{A}^{\mathrm{op}}$ is as well a k-algebroid.

The notion of k-algebra is equivalent to the classical notion of k-algebra. A k-algebroid is a k-algebra "with several objects".

Definition 2.1.6. Let $\mathcal{A}$ and $\mathcal{B}$ be two k-algebroids, a *left $\mathcal{A}$-module* (or simply $\mathcal{A}$-module) is a functor from $\mathcal{A}$ to k-mod, the category of k-modules, a *right $\mathcal{A}$-module* (or simply module-$\mathcal{A}$) is a functor from $\mathcal{A}^{\mathrm{op}}$ to k-mod. A $(\mathcal{A}, \mathcal{B})$-*bimodules* (or simply a $\mathcal{A}$-module-$\mathcal{B}$) is a functor from $\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}}$ to k-mod.

Notation 2.1.7. Let $\mathcal{A}$ be an algebroid, and M be a $\mathcal{A}$-module. If a and b are two objects of $\mathcal{A}$, we denote by ${}_a\mathcal{A}_b$ the hom-set $\hom_{\mathcal{A}}(b, a)$. An *element* of $\mathcal{A}$ is an element of one of the ${}_a\mathcal{A}_b$ for a and b two objects of $\mathcal{A}$. If $x \in {}_a\mathcal{A}_b$ and $y \in {}_b\mathcal{A}_c$, we denote by $x \cdot y$ or xy the composition $x \circ y$ (this is then an element of ${}_a\mathcal{A}_c$). If a is an object of $\mathcal{A}$, we denote ${}_aM$ the k-module $M(a)$. If m is an element of ${}_bM$, and x and an element of ${}_a\mathcal{A}_b$, then we denote by $x \cdot m$ the image of m by the k-module map $M(x)$, this is an element of ${}_aM$. We have similar notations for right modules and bimodules.

With these notations we recover the classical intuition that a $\mathcal{A}$-module is a set endowed with an action of $\mathcal{A}$.

Example 2.1.8. Let $\mathcal{A}$ be a k-algebroid, then $\mathcal{A}$ can be seen as a $\mathcal{A}$-module: an object a of $\mathcal{A}$ is sent on $\bigoplus_{b\in\mathcal{A}} {}_a\mathcal{A}_b$, an element x of ${}_a\mathcal{A}_b$ acts by composition on the left (*i.e.* by multiplication). For the same reasons, $\mathcal{A}$ can be seen as a module-$\mathcal{A}$ and as a $\mathcal{A}$-module-$\mathcal{A}$.

Definition 2.1.9. Given a $\mathcal{A}$-module M we say that N is a *sub-$\mathcal{A}$-module* of M if N is a $\mathcal{A}$-module and if for every object a of $\mathcal{A}$, ${}_aN \subset {}_aM$ and for every element x of ${}_a\mathcal{A}_b$, the map $N(x)$ is the restriction of $M(x)$ to the set ${}_bN$. We have an analogous definition for *sub-module-$\mathcal{A}$* and for *sub-bimodule*.

Definition 2.1.10. Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ be three k-algebroids, M a $\mathcal{A}_1$-module-$\mathcal{A}_2$ and N a $\mathcal{A}_2$-module-$\mathcal{A}_3$. Then we define the $\mathcal{A}_1$-module-$\mathcal{A}_3$ $M \otimes_{\mathcal{A}_2} N$, *the tensor product of M and N*:

$${}_{a_1}(M \otimes_{\mathcal{A}_2} N)_{a3} = \left(\bigoplus_{a_2 \in \mathrm{Ob}(\mathcal{A}_2)} {}_{a_1}M_{a_2} \otimes {}_{a_2}N_{a_3} \right) \Big/ {}_{a_1}I_{a_3},$$

where ${}_{a_1}I_{a_3}$ is the k-module generated by

$$\{mx \otimes n - m \otimes xn \mid x \in \mathcal{A}_2,\, m \in {}_{a_1}M,\, n \in N_{a_3}\}\,.$$

The action of $\mathcal{A}_1$ and $\mathcal{A}_3^{\mathrm{op}}$ on $M \otimes_{\mathcal{A}_2} N$ is induced by the action of $\mathcal{A}_1$ (resp. A_3^{op}) on M (resp. N).

Definition 2.1.11. Let $\mathcal{A}$ be a k-algebroid and M and N be two $\mathcal{A}$-modules, *a morphism* $\phi = (\phi_a)_{a\in \mathrm{Ob}(\mathcal{A})}$ *between* M *and* N is a natural transformation between M and N *i.e.* a collection of maps $(\phi_a : {}_aM \to {}_aN)_{a\in\mathrm{Ob}(\mathcal{A})}$ such that for every pair of objects a and b of $\mathcal{A}$ and every x in ${}_a\mathcal{A}_b$ the following diagram commutes:

$$\begin{tikzcd} {}_aN \arrow[r, "N(x)"] & {}_bN \\ {}_aM \arrow[u, "\phi_a"] \arrow[r, "M(x)"'] & {}_bM \arrow[u, "\phi_b"'] \end{tikzcd}$$

With our notations the natural transformation condition for ϕ is written:

$$\phi_a(x \cdot m) = x \cdot \phi_b(\cdot m) \quad \text{for } x \in {}_a\mathcal{A}_b \text{ and } m \in {}_bM\,.$$

We will often omit the index of ϕ, so that this last condition looks exactly like a linearity condition. When considering right modules or bimodules the notations are adapted to be natural. We can therefor form the *category* $\mathcal{A}$-mod of $\mathcal{A}$-modules. This is a k-category so in particular this is an abelian category.

Remark 2.1.12. When one has a classical k-algebra A, one can always consider the forgetful functor which goes from A-mod to k-mod. One may wonder what this functor becomes in our context. Let $\mathcal{A}$ be a k-algebroid, we will denote the forgetful functor from $\mathcal{A}$-mod to k-mod (by convention $k = k_1$ and has just one object called 1, see example 2.1.5) by For. It is defined as follows:

$$\begin{array}{rcl} M \in \mathcal{A}\text{-mod} & \longmapsto & \mathsf{For}(M) : \begin{cases} 1 & \longmapsto \bigoplus_{a\in\mathrm{Ob}(\mathcal{A})} {}_aM \\ x \in k & \longmapsto \sum_{a\in\mathrm{Ob}(\mathcal{A})} x \cdot \mathrm{id}_{{}_aM} \end{cases} \\ (\phi : M \to N) & \longmapsto & \mathsf{For}(\phi) = \sum_{a\in\mathrm{Ob}(\mathcal{A})} \phi_a \end{array}$$

In a similar way, one can define a functor from the category $\mathcal{A}$-mod-$\mathcal{B}$ to $\mathcal{A}$-mod (or mod-$\mathcal{B}$) which forget the structures of modules-$\mathcal{B}$ (or of $\mathcal{A}$-modules).

Definition 2.1.13. Let $\mathcal{A}$ and $\mathcal{B}$ be two k-algebroids, we say that $\mathcal{A}$ and $\mathcal{B}$ are *Morita equivalent* if their categories of modules are equivalent *i.e.* if there exist two functors F and G such that $F \circ G \simeq \mathrm{id}_{\mathcal{A}\text{-mod}}$ and $G \circ F \simeq \mathrm{id}_{\mathcal{B}\text{-mod}}$.

Remark 2.1.14. The Morita equivalence is, as usual, an equivalence relation.

Definition 2.1.15. Let $\mathcal{A}$ be a k-algebroid with finitely many objects indexed by a set X. We call *regularized algebra of* $\mathcal{A}$ and denote $C(\mathcal{A})$, the k-algebroid with only one object and with hom-space given by $\bigoplus_{(x,y)\in X^2} \hom_{\mathcal{A}}(x, y)$. Where the direct sum is seen in k-mod and where the composition of two un-composable morphisms in $\mathcal{A}$ is defined to be 0.

With this definition, it's easy to check that $C(\mathcal{A})$ is a k-algebroid and has only one object, so that it can be seen as a k-algebra. This construction works as well when $\mathcal{A}$ has infinitely many objects but the resulting k-algebra has no unity.

Proposition 2.1.16 ([Mit85]). *If $\mathcal{A}$ is a k-algebroid with finitely many objects then, $\mathcal{A}$ and $C(\mathcal{A})$ are Morita equivalent.*

Definition 2.1.17. Let $\mathcal{A}$ be a k-algebroid, $(a_i)_{i\in I}$ a collection of objects of $\mathcal{A}$, and $\mathbf{e} = (e_i)_{i\in I}$ a collection of idempotents (e. g. the identity morphisms of the a_i) such that for every i, e_i belongs to ${}_{a_i}\mathcal{A}_{a_i}$. Then we defined the k-algebroid $\mathcal{A}^{\mathbf{e}}$ as follows: the objects of $\mathcal{A}^{\mathbf{e}}$ are elements of I, and the hom-sets are given for every i and j in I by:

$$ {}_i(A^{\mathbf{e}})_j = e_i\, {}_{a_i}\mathcal{A}_{a_j} e_j. $$

We call this k-algebroid *the sub-algebroid of A modelled on* $\mathbf{e}$.

If $\mathcal{A}$ is a k-algebroid, and $\mathbf{e}$ like in 2.1.17 one can defined two natural bimodules: E is the $\mathcal{A}^{\mathbf{e}}$-module-$\mathcal{A}$ defined by ${}_iE_a = e_i\,{}_{a_i}\mathcal{A}_a$ and F is the $\mathcal{A}$-module-$\mathcal{A}^{\mathbf{e}}$ defined by ${}_aF_i = {}_a\mathcal{A}_{a_i}$ for every $i \in \mathrm{Ob}(\mathcal{A}^{\mathbf{e}})$ and $a \in \mathrm{Ob}(\mathcal{A})$.

Theorem 2.1.18 ([BHMV95]). *Let $\mathcal{A}$ be a k-algebroid,* $\mathbf{e}$ *like in definition 2.1.17 and E and F as before. Suppose that the collection* $\mathbf{e}$ *of idempotents generates $\mathcal{A}$ as a $\mathcal{A}$-module-$\mathcal{A}$, then the $\mathcal{A}^{\mathbf{e}}$-module-$\mathcal{A}^{\mathbf{e}}$ $E \otimes_{\mathcal{A}} F$ is isomorphic to $\mathcal{A}^{\mathbf{e}}$ and the $\mathcal{A}$-module-$\mathcal{A}$ $F \otimes_{\mathcal{A}^{\mathbf{e}}} E$ is isomorphic to $\mathcal{A}$. Consequently $\mathcal{A}$ and $\mathcal{A}^{\mathbf{e}}$ are Morita equivalent.*

2.1.2 Algebroids revisited

In this subsection we give a very elementary approach to algebroids. We redefined the terms algebroids, modules, etc. This is not quite standard but this is a down-to-earth presentation of these notions. We will restate and prove the results given in 2.1.1. In this section all algebras are associative, and by non-unital, we mean "possibly non-unital".

Definition 2.1.19. Let I be a set, and let $(\mathcal{A}, \cdot_{\mathcal{A}})$ be a non-unital k-algebra. We say that $\mathcal{A}$ is *k-algebroid (with objects in I)* if there exists a decomposition of $\mathcal{A}$ as a direct sum:

$$ \mathcal{A} = \bigoplus_{i,j\in I} {}_i\mathcal{A}_j $$

such that:

(i) for every i, j, p and q in I, ${}_i\mathcal{A}_j \cdot {}_p\mathcal{A}_q = \{0\}$ if $j \neq p$;

(ii) for every i, j and p in I, ${}_i\mathcal{A}_j \cdot {}_j\mathcal{A}_p \subseteq {}_i\mathcal{A}_p$;

(iii) for every i in I, there exists an element $1_i \in {}_i\mathcal{A}_i$ such that:

$$ \forall j \in I, \forall x \in {}_i\mathcal{A}_j,\ 1_i \cdot x = x \quad \text{and} \quad \forall j \in I, \forall x \in {}_j\mathcal{A}_i,\ x \cdot 1_i = x. $$

In the following we denote I by $\mathrm{Ob}(\mathcal{A})$, and for $i \in \mathrm{Ob}(\mathcal{A})$ we write ${}_i\mathcal{A}$ for $\bigoplus_{j\in I} {}_i\mathcal{A}_j$ and $\mathcal{A}_i$ for $\bigoplus_{j\in I} {}_j\mathcal{A}_i$.

Note that the previous definition implies that the elements 1_i (for i in $\mathrm{Ob}(\mathcal{A})$) are unique.

Example 2.1.20. Unital k-algebras can be seen as k-algebroids with one object. If I is a set and $(\mathcal{A}^i)_{i\in I}$ is a collection of k-algebras, then the direct sum of the $\mathcal{A}^i$ is a k-algebroids with object in I by setting ${}_i\mathcal{A}_i = \mathcal{A}^i$ and ${}_i\mathcal{A}_j = \{0\}$ for $i \neq j$. Note that if $\mathcal{A}$ is a k-algebroid, $\mathcal{A}^{\mathrm{op}}$, the opposite (non-unital) k-algebra can be given a structure of k-algebroid with the same set of objects by setting ${}_i(\mathcal{A}^{\mathrm{op}})_j = {}_j\mathcal{A}_i$ for all i and j in $\mathrm{Ob}(\mathcal{A}) = \mathrm{Ob}(\mathcal{A}^{\mathrm{op}})$.

Definition 2.1.21. Let $\mathcal{A}$ be a k-algebroid, and M be a module over the underlying (non-unital) algebra $\mathcal{A}$. We say that *M is a left module over the k-algebroid $\mathcal{A}$ (or simply a $\mathcal{A}$-module)* if we have the following decomposition of M as a k-module:

$$M = \bigoplus_{i\in\mathrm{Ob}(\mathcal{A})} 1_i M. \tag{2.1}$$

We denote the 1_iM by ${}_iM$. Note that if $i \neq j$, $\mathcal{A}_i \cdot {}_jM = \{0\}$, and that ${}_j\mathcal{A}_i \cdot {}_iM \subseteq {}_jM$. Similarly we define a *right module* and a *bimodule.*

Remark 2.1.22. Note that the fact that the sum is direct follows from the k-algebroid structure of $\mathcal{A}$, hence the only non-trivial requirement for M is that $M \subseteq \sum_{i\in\mathrm{Ob}(\mathcal{A})} {}_iM$.

Example 2.1.23. A k-algebroid $\mathcal{A}$ can be seen as a left module, a right module or bi-module on itself. If M and N are two $\mathcal{A}$-modules, then $M \oplus N$ is a $\mathcal{A}$-module.

Definition 2.1.24. Let $\mathcal{A}$ be an k-algebroid, M and N be two $\mathcal{A}$-modules, and φ be a $\mathcal{A}$-linear map from M to N (here $\mathcal{A}$ is seen as non-unital algebra). The map φ is *a $\mathcal{A}$-module map* if for all i in $\mathrm{Ob}(\mathcal{A})$, $\varphi({}_iM) \subseteq {}_iN$. A $\mathcal{A}$-module map φ a sum of maps $(\varphi_i)_{i\in Ob(\mathcal{A})}$ with $\varphi_i : {}_iM \to {}_iN$ for all i in $\mathrm{Ob}(\mathcal{A})$. Given a k-algebroid $\mathcal{A}$ one can consider the category of $\mathcal{A}$-modules, whose objects are $\mathcal{A}$-modules and morphisms are $\mathcal{A}$-module maps.

In the definition of $\mathcal{A}$-modules maps, the last condition is always satisfied, since it comes from the $\mathcal{A}$-linearity of the map φ. This gives us the following lemma:

Lemma 2.1.25. *Let $\mathcal{A}$ be a k-algebroid* , the category $\mathcal{A}\text{-}\mathsf{mod}$ of $\mathcal{A}$-modules *is a full sub-category of the category of module over the underlying non-unital algebra $\mathcal{A}$.*

Definition 2.1.26. The tensor product over $\mathcal{A}$ is the tensor product over $\mathcal{A}$ as a non-unital algebra. With this tensor product the category $\mathcal{A}\text{-}\mathsf{mod}$ becomes a monoidal category. Two k-algebroids are *Morita equivalent* when their categories of modules are equivalent.

Remark 2.1.27. (a) For $\mathcal{A}$ a non-unital algebra and M a $\mathcal{A}$-module, there is no reason that $\mathcal{A} \otimes_{\mathcal{A}} M \simeq M$ as a $\mathcal{A}$-module, however, this holds for algebroids.

(b) The algebroid $\hom_{\mathcal{A}\text{-}\mathsf{mod}}(\mathcal{A}, \mathcal{A})$ needs not to be isomorphic to $\mathcal{A}$, one explanation for this is that while the k-algebra of endomorphisms is unital (via $\mathrm{id}_{\mathcal{A}}$), the k-algebra $\mathcal{A}$ may not be unital.

(c) The Morita equivalence is an equivalence relation.

Definition 2.1.28. Let $\mathcal{A}$ be a k-algebroid such that $\mathrm{Ob}(\mathcal{A})$ is finite, then the underlying non-unital algebra is actually unital, and we denote this algebra by $C(\mathcal{A})$ and call it *the regularised algebra of $\mathcal{A}$.*

Proposition 2.1.29. *If $\mathcal{A}$ is a k-algebroid with finitely many objects, then $\mathcal{A}$ and $C(\mathcal{A})$ are Morita equivalent.*

Proof. This is actually obvious because $\mathcal{A}$-mod is by definition the full subcategory of $C(\mathcal{A})$-mod with object satisfying relation (2.1). But every $C(\mathcal{A})$-mod satisfies this relation. So that $\mathcal{A}$-mod and $C(\mathcal{A})$-mod are the same category. □

The end of this section is devoted to prove theorem 2.1.18 in our new context. Let $\mathcal{A}$ be a k-algebroid and I a subset of $\mathrm{Ob}(\mathcal{A})$. For all $i \in I$, let us choose e_i an idempotent element of ${}_i\mathcal{A}_i$. Let us consider $\mathcal{B}$ the k-algebroid (with objects in I) given by:

$$ {}_i\mathcal{B}_j = e_i \cdot {}_i\mathcal{A}_j \cdot e_j. $$

Suppose now that $\mathcal{A}$, as a $\mathcal{A}$-module-$\mathcal{A}$, is generated by the family $(e_i)_{i\in I}$. We want to show that in this situation, the k-algebroids $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent. We consider E the sub-$\mathcal{B}$-module-$\mathcal{A}$ of $\mathcal{A}$ and F the sub $\mathcal{A}$-module-$\mathcal{B}$ of $\mathcal{A}$, given by:

$$ {}_iE_a = e_i \cdot {}_i\mathcal{A}_a \quad \text{and} \quad {}_aF_i = {}_a\mathcal{A}_i \cdot e_i. $$

We claim that $E \otimes_{\mathcal{A}} F \simeq \mathcal{B}$ as a $\mathcal{B}$-module-$\mathcal{B}$ and $F \otimes_{\mathcal{B}} E \simeq \mathcal{A}$ as a $\mathcal{A}$-module-$\mathcal{A}$. The isomorphisms are given by:

$$ \begin{array}{rccc} \varphi : & E \otimes_{\mathcal{A}} F & \longrightarrow & \mathcal{B} \\ & e_i x \otimes y e_j & \longmapsto & e_i x y e_j, \end{array} \quad \text{and} \quad \begin{array}{rccc} \psi : & F \otimes_{\mathcal{B}} E & \longrightarrow & \mathcal{A}, \\ & x e_i \otimes e_i y & \longmapsto & x e_i y. \end{array} $$

The fact that φ is an isomorphism is clear, for ψ, this is a little less obvious. We know that $\mathcal{A}$ is generated by the family $(e_i)_{i\in I}$ as a $\mathcal{A}$-module-$\mathcal{A}$, hence for every object a of $\mathcal{A}$, we can express 1_a as a finite sum:

$$ 1_a = \sum_i x_i^{(a)} e_i y_i^{(a)}, $$

where $x_i^{(a)}$ belongs to ${}_a\mathcal{A}_i$ and $y_i^{(a)}$ belongs to ${}_i\mathcal{A}_a$. Now let us define ξ:

$$ \begin{array}{rccl} \xi : & \mathcal{A} & \longrightarrow & F \otimes_{\mathcal{B}} E, \\ & {}_a\mathcal{A}_b \ni z & \longmapsto & \sum_i x_i^{(a)} e_i \otimes e_i y_i^{(a)} z = \sum_j z x_j^{(b)} e_j \otimes e_j y_j^{(b)} \end{array} $$

The maps ξ and ψ are mutually inverse, so that ψ is an isomorphism. And this proves that the functors $E \otimes_{\mathcal{A}} \_$ and $F \otimes_{\mathcal{B}} \_$ gives an equivalence of categories between $\mathcal{A}$-mod and $\mathcal{B}$-mod. For this last step, it is important that $\mathcal{A}$ and $\mathcal{B}$ are algebroids (see remark 2.1.27).

Definition 2.1.30. The 2-category k-Aloid is given by the following data:

- 0-objects: k-algebroids,
- 1-morphisms from $\mathcal{A}_1$ to $\mathcal{A}_0$: $\mathcal{A}_0$-module-$\mathcal{A}_1$,
- 2-morphisms between two $\mathcal{A}_0$-modules-$\mathcal{A}_1$ M and N: maps of $\mathcal{A}_0$-module-$\mathcal{A}_1$ between M and N.

The composition of 1-morphisms is given by the tensor product over the appropriate algebroid and the identity morphism of a k-algebroid $\mathcal{A}$ is $\mathcal{A}$ seen as a $\mathcal{A}$-module-$\mathcal{A}$ (see remark 2.1.27).

2.2 A $\mathfrak{sl}_3$-TQFT with corners

The aim of this part is to extend the (webs,foams)-TQFT $\mathcal{F}$ defined in 1.2 to web tangles and "foams with corners". We want to recover the functor $\mathcal{F}$ when a web tangle is a closed web, and when a foam has no corner. Furthermore we want to have a gluing formula, *i.e.* if w_0 and w_1 are two composable web tangles we want that $\mathcal{F}(w_0w_1)$ to be obtained by "gluing" $F(w_0)$ and $F(w_1)$ (see lemma 2.2.16). This last requirement will be satisfied if our extension is a 2-functor. To do this, we follow the strategy of [Kho02]. In the sequel R will be a commutative ring (the same as in chapter 1).

2.2.1 The 2-category $\mathfrak{sl}_3$-2-Foam

A 2-functor goes from a 2-category to another 2-category. We give details here of the source 2-category of "foams with corners".

Definition 2.2.1. Let ε_0 and ε_1 be two sequences of signs. If w_b and w_t are $(\varepsilon_0, \varepsilon_1)$-web tangles, a (w_b, w_t)*-foam* f is the intersection of a foam f' with $\mathbb{R} \times [0,1] \times [0,1]$ such that:

- there exists $\eta_0 \in]0,1]$ such that $f \cap \mathbb{R} \times [0,\eta_0] \times [0,1] = \{1,2,\dots,l(\varepsilon_0)\} \times [0,\eta_0] \times [0,1]$,
- there exists $\eta_1 \in [0,1[$ such that $f \cap \mathbb{R} \times [\eta_1,1] \times [0,1] = \{1,2,\dots,l(\varepsilon_1)\} \times [\eta_1,1] \times [0,1]$,
- there exists $\eta_b \in]0,1]$ such that $f \cap \mathbb{R} \times [0,1] \times [0,\eta_b] = w_b \times [0,\eta_b]$,
- there exists $\eta_t \in [0,1[$ such that $f \cap \mathbb{R} \times [0,1] \times [\eta_t,1] = w_t \times [\eta_t,1]$.

We require as well the compatibility of the orientations of the facets of f with the reversed orientations of w_b and with the orientations of w_t (see figure 2.1). If f_b is a (w_b, w_m)-foam

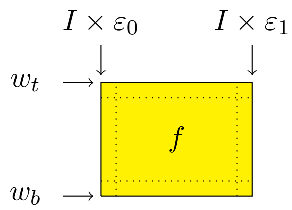


Figure 2.1: A (w_b, w_t)-foam, the small parts located between the boundary and the dotted lines are cartesian products of the boundary times a small interval.

and f_t is a (w_m, w_t)-foam we define $f_t \circ f_b$ *the composition of* f_b *and* f_t to be the (w_b, w_t)-foam obtained by gluing f_b and f_t along w_m and resizing (see figure 2.2). We define $\deg f$,

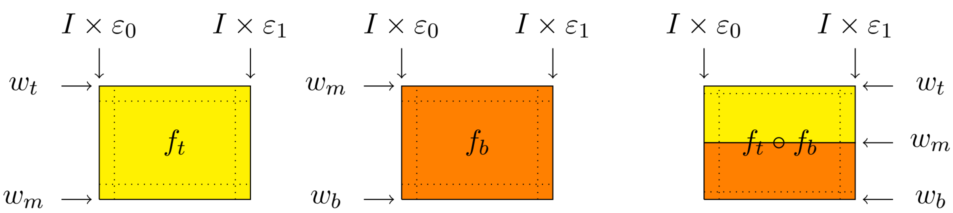


Figure 2.2: The composition of f_b and f_t.

the *degree* of a (w_b, w_t)-foam f by the following formula:

$$\deg(f) = -2\chi(\widetilde{f}) + \chi(w_b) + \chi(w_t),$$

where $\widetilde{f}$ is the foam f with the dots on its facets punctured out.

Definition 2.2.2. The 2-category $\mathfrak{sl}_3$-2-Foam is the (non-strict) 2-category with:

- 0-objects: admissible sequences of signs;
- 1-morphisms between ε_0 and ε_1: pairs (w, n) with w a $(\varepsilon_0, \varepsilon_1)$-web tangle, and n an integer;
- 2-morphisms between (w_b, n_b) and (w_t, n_t): R-linear combinations of (w_b, w_t)-foams up to ambient isotopy relative to the boundary.

The 1-morphisms $(w, 0)$ will often be denoted by w. The 2-morphisms spaces are graded: if α is a 2-morphism between (w_b, n_b) and (w_t, n_t) represented by a (w_b, w_t)-foam f, then:

$$\deg \alpha = \deg f - n_t + n_b,$$

where $\deg f$ is the degree of f as a (w_b, w_t)-foam.

Remark 2.2.3. The term “non-strict” means that the composition of 1-morphisms is not associative but associative up to a unique 2-isomorphism. The restriction to admissible sequences of signs will become clear in the following.

Proposition 2.2.4. *Let ε be an admissible sequence of signs, and w_b and w_t two ε-webs. Let us denote by $\widetilde{V}$ the free graded R-module generated by isotopy classes of (w_t, w_t)-foams, and by V the graded R-module $\widetilde{V}$ modded out by the foam relations **FR**. Then V is a free R-module and its graded dimension is equal to $\langle \overline{w_b} w_t \rangle \cdot q^{l(\varepsilon)}$.*

Proof. This is actually a corollary from the theorem 1.2.10. Indeed, the graded module V is isomorphic to $\mathcal{F}(\overline{w_b} w_t)$ up to a grading shift (the way to define the degree differs a little): both modules are spanned by foams with certain boundary conditions and mod out by the foam relations **FR**, the only difference is that for V the boundary condition is expressed in a different way. The figure 2.3 gives an illustration of the correspondence between $\mathcal{F}(w)$ and V.

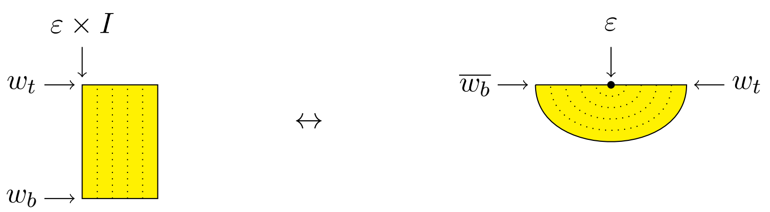


Figure 2.3: Isomorphism between V (on the left) and $\mathcal{F}(\overline{w_b} w_t)$ (on the right) at the level of foams.

What remains to check is the degree shift. It’s enough to compare $\chi(w_b) + \chi(w_t)$ and $\chi(\overline{w_b} w_t)$. We have:

$$\chi(\overline{w_b} w_t) = \chi(w_b) + \chi(w_t) - l(\varepsilon),$$

because to “connect” $\overline{w_b}$ and w_t (in order to obtain $\overline{w_b} w_t$) one just need to add $l(\varepsilon)$ edges between the univalent vertices of $\overline{w_b}$ and the univalent vertices of w_t. Hence we have $\dim_q V = \langle \overline{w_b} w_t \rangle \cdot q^{l(\varepsilon)}$. $\square$

	$\mathfrak{sl}_3$-2-Foam	R-Aloid	ref.
0-objects	admissible sequences of signs	R-algebroids	2.2.7
1-morphisms	web tangles	bimodules	2.2.11
2-morphisms	$(\cdot,\cdot)$-foams	bimodules maps	2.2.13
composition of 1-morphisms	gluing web tangles	tensor products	2.2.16
composition of 2-morphisms	stacking $(\cdot,\cdot)$-foams	compositions of maps	2.2.15

Table 2.1: The 2-functor dictionary.

What we will construct in the rest of this chapter is the a 2-functor from the 2-category $\mathfrak{sl}_3$-2-Foam into the 2-category of R-Aloid. The following table sum up the expectations:

In order to define easily the 2-functor we need other definitions. This is still about foams but we need other boundary conditions for these foams.

Definition 2.2.5. Let ε_0 and ε_1 be two admissible sequences of signs, and let w_i be an ε_i-web and w_m be a $(\varepsilon_0, \varepsilon_1)$-web tangle. *A (w_0, w_m, w_1)-vfoam f* is the intersection of a foam f' with $\mathbb{R} \times [0,1] \times [0,1]$ such that:

- there exists $\eta_0 \in]0,1]$ such that $f \cap \mathbb{R} \times [0,\eta_0] \times [0,1] =$ "$w_0 \times [0,\eta_0]$" $= \{(x,y,z) \in \mathbb{R} \times [0,\eta_0] \times [0,1] | (x,z) \in w_0\}$,
- there exists $\eta_1 \in [0,1[$ such that $f \cap \mathbb{R} \times [\eta_1,1] \times [0,1] =$ "$w_1 \times [\eta_1,1]$" $= \{(x,y,z) \in \mathbb{R} \times [\eta_1,1] \times [0,1] | (x,z) \in w_1\}$,
- there exists $\eta_b \in]0,1]$ such that $f \cap \mathbb{R} \times [0,1] \times [0,\eta_b] = \emptyset$,
- there exists $\eta_t \in [0,1[$ such that $f \cap \mathbb{R} \times [0,1] \times [\eta_t,1] = w_m \times [\eta_t,1]$.

The condition are sum up on figure 2.4). Furthermore, we ask that the orientations of the facets of f to be compatible with the reversed orientations of w_0 and the orientations of w_m and w_1. If $\varepsilon_0 = \varepsilon_1$ and $w_m = \varepsilon_0 \times [0,1]$ then we will speak of $(w_0, \varepsilon_0, w_1)$-vfoam. One

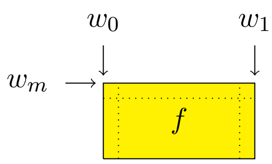


Figure 2.4: A (w_0, w_m, w_1)-vfoam, the small part located between the boundary and the dotted lines are cartesian products.

can glue a (w_0, w_m, w_1)-vfoam f with a (w_1, w_t, w_2)-vfoam g along w_1 and after resizing we obtain fg a $(w_0, w_m w_t, w_2)$-vfoam (see figure2.5). We call this operation *multiplication*. Note that when one composes a (w_0, ε, w_1)-vfoam and a (w_1, ε, w_2)-vfoam, one obtain a (w_0, ε, w_2)-vfoam.

2.2.2 The algebroid $\mathbb{K}^\varepsilon$

The main ingredients of the construction of the TQFT with corners are the algebras (or algebroids) associated with 0-objects, here admissible sequences of signs. During the construction we should focus on the idea that we want a gluing formula at the end. Hence the algebroids associated to ε should "contain" all the ways to complete ε on both sides to obtain a closed web. The remark lead us naturally to defined the algebroids $\widetilde{K^\varepsilon}$ as follows:

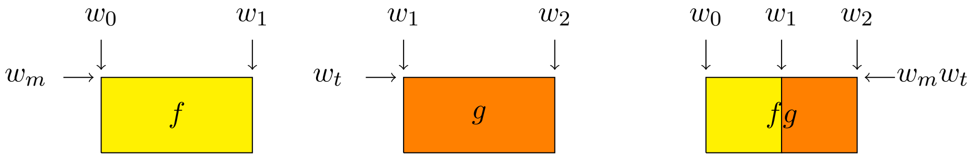


Figure 2.5: Multiplication of a (w_0, w_m, w_1)-vfoam f with a (w_1, w_t, w_2)-vfoam g.

Definition 2.2.6. Let ε be an admissible sequence of signs, we define *the algebroid* $\widetilde{K^\varepsilon}$ to be the R-category with objects ε-webs and with morphisms from w_0 to w_1 given by R-linear combination of (w_0, ε, w_1)-vfoams up to isotopy and up to the foam relations **FR** (see proposition 1.2.11). If we want to take the point of view of section 2.1.2 we would say that $\widetilde{K^\varepsilon}$ is the free R-module spanned by all the (w_0, w_1)-vfoams with w_0 and w_1 two ε-webs subjected to the foam relations **FR** and with product given by composition whenever this is possible and by 0 otherwise. The objects of $\mathbb{K}^\varepsilon$ are ε-webs.

The algebroids $\widetilde{K^\varepsilon}$ seem to be the right structures to consider, but they have the disadvantage to be highly infinite dimensional. This is why we consider the R-algebroids $\mathbb{K}^\varepsilon$ which are much smaller algebroids. These algebroids will be Morita equivalent to the previous ones thanks to theorem 2.1.18. Note that the admissibility condition on ε ensures that the algebroid $\widetilde{K^\varepsilon}$ has a non empty set of objects.

Definition 2.2.7. Let ε be an admissible sequence of signs. For each isotopy class of non-elliptic ε-web we choose one base point and we consider W^ε the (finite) set of all these base points. We are now ready to define $\mathbb{K}^\varepsilon$: this is the full sub-category of $\widetilde{K^\varepsilon}$ where we require the objects to belong to W^ε. With the other point of view, we define K^ε to be the R-vector space spanned by all the (w_0, ε, w_1)-vfoams with w_0 and w_1 two elements of W^ε subjected to the foam relations **FR** and with product given by composition whenever this is possible and by 0 otherwise. We denote by K^ε the algebra $C(\mathbb{K}^\varepsilon)$ the regularized algebra of $\mathbb{K}^\varepsilon$.

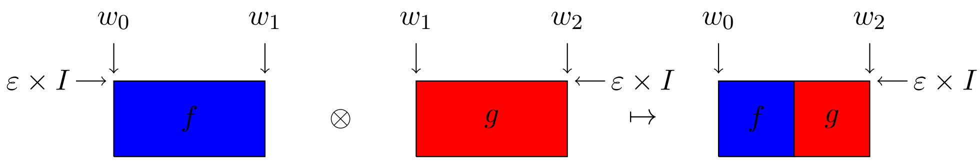


Figure 2.6: The product in $\mathbb{K}^\varepsilon$ when the compatibiltiy holds (else the product is defined to be 0).

From proposition 1.1.23 and theorem 1.2.10 we can see that for all ε, $\mathbb{K}^\varepsilon$ is a finite dimensional algebra.

Proposition 2.2.8. *For all admissible sequences of signs, the algebroids* $\widetilde{K^\varepsilon}$ *and* $\mathbb{K}^\varepsilon$ *are Morita equivalent. Consequently* $\widetilde{K^\varepsilon}$ *and* K^ε *are Morita equivalent.*

Proof. In view of the theorem 2.1.18 it is enough to show that the collection $(\mathrm{id}_w)_{w\in W^\varepsilon}$ generate $\widetilde{K^\varepsilon}$ as a $\widetilde{K^\varepsilon}$-module-$\widetilde{K^\varepsilon}$. It is clear the collection $(\mathrm{id}_v)_v$ where v runs trough all ε-webs generate $\widetilde{K^\varepsilon}$ as a $\widetilde{K^\varepsilon}$-module-$\widetilde{K^\varepsilon}$. So we just need to show that for every ε-web v, id_v can be written as a finite sum of terms of $\widetilde{K^\varepsilon}$, each of these terms admitting one element of $(\mathrm{id}_w)_{w\in W^\varepsilon}$ as a factor. If v is non-elliptic it is isotopic to a web w of W^ε and then the statement is obvious, if it is not then we can apply the square relation, the digon relations (see figure 1.11) and the surgery formula (see figure 1.9) so that id_v factors through a finite sum of id_w with w in W^ε. □

Remark 2.2.9. The second part of the statement is a direct application of proposition 2.1.29. In fact K^ε is equal to K^ε where we have forgotten the "oid"-structure of the algebroid. For this reason all the construction done in this section and in the following section could have been done over K^ε instead of $\mathbb{K}^\varepsilon$, the results and the proofs remain true. In [MPT12], they deal with the algebra K^ε.

2.2.3 Bimodules and bimodules maps.

Now that we have defined the algebroids $\mathbb{K}^\varepsilon$, we should define the bimodules associated with web tangles.

Definition 2.2.10. Let ε_0 and ε_1 be two admissible sequences of signs, w_i and w_i' be elements of W^{ε_i} for $i = 0, 1$, and w be a $(\varepsilon_0, \varepsilon_1)$-web tangle. We consider f a (w_0, w, w_1)-vfoam, a a $(w_0', \varepsilon_0, w_0)$-vfoam and b a $(w_1, \varepsilon_1, w_1')$-vfoam. We define the (w_0', w, w_1)-vfoam af. Let $\eta_0 \in]0, 1]$ such that

$$f \cap \mathbb{R} \times [0, \eta_0] \times [0, 1] = \text{"}w_0 \times [0, \eta_0]\text{"} = \{(x, y, z) \in \mathbb{R} \times [0, \eta_0] \times [0, 1] | (x, z) \in w_0\},$$

then we set

$$af = \left\{(x, y, z) \in \mathbb{R} \times [0, \eta_0] \times [0, 1] \middle| \left(x, \frac{y}{\eta_0}, z\right) \in a\right\} \cup (\mathbb{R} \times [\eta_0, 1] \times [0, 1] \cap f)\,.$$

with orientations and dots on facets induced by a and f. This is not completely well defined because, we make the choice of η_0, however the isotopy class (relatively to the boundary) of af is well-defined and is invariant under isotopy of a and of f. The (isotopy class of) (w_0, w, w_1')-vfoam fb and the (isotopy class of) (w_0', w, w_1')-vfoam afb are defined in the same way. The definition of afb is illustrated on figure 2.7.

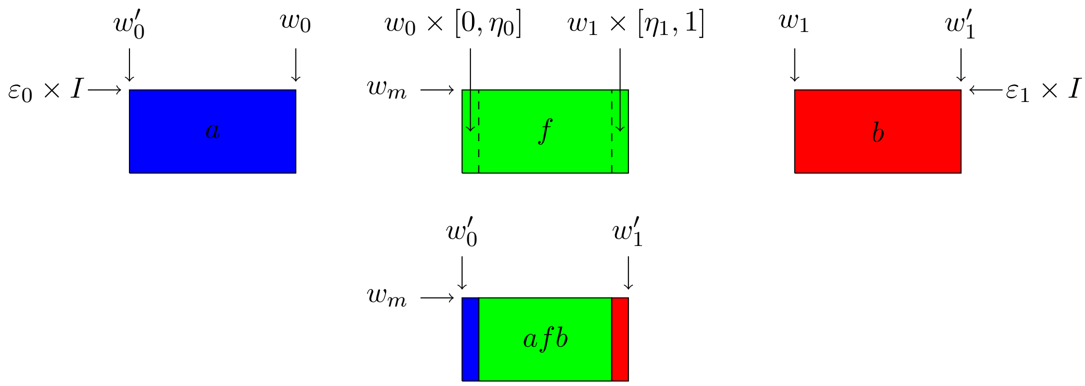


Figure 2.7: Illustration of the construction of afb. The dashed line are meant to be the lines along which one should cut f along.

With the same notations, we have $(af)b = a(fb) = afb$, furthermore, for a' and b' compatible vfoams, we have $(a'a)f = a'(af)$ and $f(bb') = (fb)b'$ (up to isotopy relative to the boundary).

Definition 2.2.11. Let ε_0 and ε_1 be two admissible sequences of signs and w a $(\varepsilon_0, \varepsilon_1)$-web tangle, we define $\mathcal{F}((w,0))$ to be the $\mathbb{K}^{\varepsilon_0}$-module-$\mathbb{K}^{\varepsilon_1}$ described by the following data: if w_i is in W^{ε_i} for $i = 0, 1$, ${}_{w_0}\mathcal{F}(w)_{w_1}$ is the graded R-vector space spanned by (w_0, w, w_1)-vfoam up to isotopy and up to the foam relations **FR**. The $\mathbb{K}^{\varepsilon_0}$-module-$\mathbb{K}^{\varepsilon_1}$ structure is given by the construction of definition 2.2.10. The $\mathbb{K}^{\varepsilon_0}$-module-$\mathbb{K}^{\varepsilon_1}$ $\mathcal{F}((w,n))$ is equal to $\mathcal{F}((w,0))$ with degree shifted by n.

Now that we have defined the bimodules associated with web tangles the next step is to define bimodule maps associated with (w_b, w_t)-foams. We will check that all our definitions give together a 2-functor afterwards.

Definition 2.2.12. Let ε_0 and ε_1 be two admissible sequences of signs, w_i be elements of W^{ε_i} for $i = 0, 1$, and w_b and w_t be two $(\varepsilon_0, \varepsilon_1)$-web tangles. Let f be a (w_0, w_b, w_1)-vfoam and u be (w_b, w_t)-foam. Let η_t in $]0, 1]$ such that:

$$f \cap \mathbb{R} \times [0,1] \times [\eta_t, 1] = w_b \times [\eta_t, 1].$$

Then we set

$$^{u}f = \left\{ (x,y,z) \in \mathbb{R} \times [0,1] \times [\eta_t, 1] \,\middle|\, \left(x, y, \frac{z-\eta_t}{1-\eta_t}\right) \in u \right\} \cup \left(\mathbb{R} \times [0,1] \times [0, \eta_t] \cap f\right),$$

with orientations and dots on facets induced by u and f. As before, this is well-defined only up to isotopy relatively to the boundary. This is illustrated on figure 2.8.

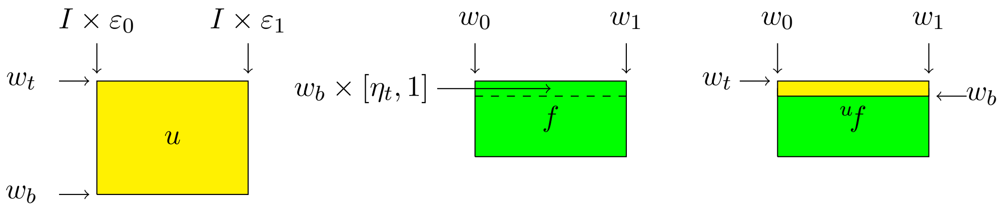


Figure 2.8: Definition of the vfoam ^{u}f.

Note that for f a vfoam and u and v compatible foams, we have $^{(u \circ v)}f = ^{u}(^{v}f)$ (up to isopoty relatively to the boundary).

Definition 2.2.13. Let ε_0 and ε_1 be two admissible sequences of signs, w_b and w_t two $(\varepsilon_0, \varepsilon_1)$-web tangles and u a (w_b, w_t)-foam. The linear map $\mathcal{F}(u) : \mathcal{F}(w_b) \to \mathcal{F}(w_t)$, is determined by the following collection of maps:

$${}_{w_0}\mathcal{F}(u)_{w_1} : {}_{w_0}\mathcal{F}(w_t)_{w_1} \longrightarrow {}_{w_0}\mathcal{F}(w_t)_{w_1},$$

for w_i in W^{ε_i} for $i = 0, 1$. If $\overline{f}$ is an element of ${}_{w_0}\mathcal{F}(w_b)_{w_1}$ represented by a (w_0, w_b, w_1)-vfoam f we define that ${}_{w_0}\mathcal{F}(u)_{w_1}(\overline{f})$ is the element of ${}_{w_0}\mathcal{F}(w_t)_{w_1}$ represented by ^{u}f.

Proposition 2.2.14. *Let ε_0 and ε_1 be two admissible sequences of signs, w_b and w_t be two $(\varepsilon_0, \varepsilon_1)$-web tangles and u be a (w_b, w_t)-foam. Then the map $\mathcal{F}(u) : \mathcal{F}(w_b) \to \mathcal{F}(w_t)$ is a $\mathbb{K}^{\varepsilon_0}$-module-$\mathbb{K}^{\varepsilon_1}$ map.*

Proof. We only need to check that the action of $\mathbb{K}^{\varepsilon_0}$ and $\mathbb{K}^{\varepsilon_1}$ commutes with the application of $\mathcal{F}(u)$. This is due to the fact that the modifications of the vfoams corresponding to the map $\mathcal{F}(u)$ and to the actions of the algebroids take place on (almost) disjoint locations, this

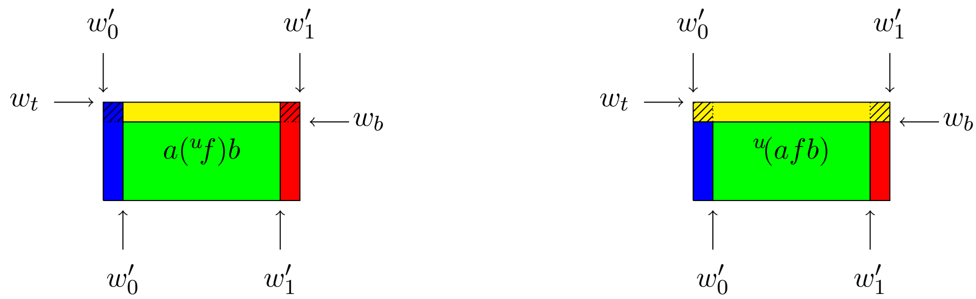


Figure 2.9: The map $\mathcal{F}(u)$ is a bimodule map, in fact, a (w_0, w_t, w_1)-vfoam representing ${}_{w_0}\mathcal{F}(u)_{w_1}(afb)$ is isotopic to a (w_0, w_t, w_1)-vfoam representing $a({}_{w_0}\mathcal{F}(u)_{w_1}(f))b$.

is illustrated on figure 2.9: A priori, the only difference between the two pictures occurs at the two hashed corners. But looking back to the definitions of afb and ${}^u f$, it appears that in these corners in both case the vfoam is equal to ε_0 or ε_1 times a small square, hence the two pictures are actually the same, and this proves that $\mathcal{F}(u)$ is a bimodule map. □

From the definition, it is clear that if u and v are two isotopic (w_m, w_t)-foams relatively to the boundary, then the bimodule maps $\mathcal{F}(u)$ and $\mathcal{F}(v)$ are equal.

Lemma 2.2.15. *Let w_m and w_n be two $(\varepsilon_0, \varepsilon_1)$ web tangles with ε_i admissible sequences of signs for $i = 0$, 1.*

1) If w_m and w_n are isotopic (relatively to the boundary), then $\mathcal{F}(w_m)$ and $\mathcal{F}(w_n)$ are isomorphic.

2) The identity (w_m, w_m)-foam (i.e. $[0,1] \times w_m$) is sent by $\mathcal{F}$ on $\mathrm{id}_{\mathcal{F}(w_m)}$.

3) If u and v are compatible foams, the bimodules maps $\mathcal{F}(u \circ v)$ and $\mathcal{F}(u) \circ \mathcal{F}(v)$ are equal.

Proof. We prove 1). As the isotopy between w_m and w_n is relative to the boundary, it provides u a (w_m, w_n)-foam when read in one direction and v a (w_n, w_m)-foam when read in the other direction. As $u \circ v$ is isotopic to $[0,1] \times w_m$ and $u \circ v$ is isotopic to $[0,1] \times w_n$, this shows that $\mathcal{F}(u)$ and $\mathcal{F}(v)$ are isomorphisms between $\mathcal{F}(w_m)$ and $\mathcal{F}(w_n)$. The statements 2 and 3 follows from the definitions. □

Lemma 2.2.16. *Let ε_i for $i = 0$, 1, 2 be admissible sequences of signs, and let w_{m_1} and w_{m_2} be respectively a $(\varepsilon_0, \varepsilon_1)$-web tangle and a $(\varepsilon_1, \varepsilon_2)$-web tangle, then $\mathcal{F}(w_{m_1} w_{m_2}) \simeq \mathcal{F}(w_{m_1}) \otimes_{\mathbb{K}^{\varepsilon_1}} \mathcal{F}(w_{m_2})$ as $\mathbb{K}^{\varepsilon_0}$-modules-$\mathbb{K}^{\varepsilon_2}$.*

Proof. One can first define the morphism

$$\begin{array}{rccl} \widetilde{\varphi}: & \mathcal{F}(w_{m_1}) \otimes_R \mathcal{F}(w_{m_2}) & \longrightarrow & \mathcal{F}(w_{m_1} w_{m_2}) \\ & f \otimes g & \longmapsto & \begin{cases} fg & \text{if } f \text{ and } g \text{ are compatible,} \\ 0 & \text{else,} \end{cases} \end{array}$$

where fg is defined for compatible vfoams at the end of the definition 2.2.5. The fact that φ is a bimodule maps comes from the same geometric argument used in the proof of proposition 2.2.14 (see figure 2.9). If a is a compatible $(w_0, \varepsilon_1, w_1)$-vfoam (with $w_i \in W^{\varepsilon_i}$ for

$i = 0, 1$), then $f(ag)$ and $(fa)g$ are isotopic. This observation shows that $\widetilde{\varphi}$ factorise through $\mathcal{F}(w_{m_1}) \otimes_{\mathbb{K}^{\varepsilon_1}} \mathcal{F}(w_{m_2})$. We denote by φ the induced morphism from $\mathcal{F}(w_{m_1}) \otimes_{\mathbb{K}^{\varepsilon_1}} \mathcal{F}(w_{m_2})$ to $\mathcal{F}(w_{m_1}w_{m_2})$. We claim that φ is an isomorphism. We first use the algebroid structures of $\mathbb{K}^{\varepsilon_0}$ and $\mathbb{K}^{\varepsilon_2}$ to cut φ into small parts:

$$\varphi = \sum_{\substack{w_0 \in W^{\varepsilon_0} \\ w_2 \in W^{\varepsilon_2}}} {}_{w_0}\varphi_{w_2} \quad \text{with} \quad {}_{w_0}\varphi_{w_2} : {}_{w_0}\mathcal{F}(w_{m_1}) \otimes_{\mathbb{K}^{\varepsilon_1}} \mathcal{F}(w_{m_2})_{w_2} \longrightarrow {}_{w_0}\mathcal{F}(w_{m_1}w_{m_2})_{w_2}.$$

It's enough to prove that ${}_{w_0}\varphi_{w_2}$ is an isomorphism for every (w_0, w_2) in $W^{\varepsilon_0} \times W^{\varepsilon_2}$. We show that these maps are both injective and surjective. For the rest of the proof we set (w_0, w_2) to be an element of $W^{\varepsilon_0} \times W^{\varepsilon_2}$.

Surjectivity: The map ${}_{w_0}\varphi_{w_2}$ being R-linear it's enough to show that if an element $\overline{f}$ of ${}_{w_0}\mathcal{F}(w_{m_1}w_{m_2})_{w_2}$ is represented by a $(w_0, w_{m_1}w_{m_2}, w_2)$-vfoam f, then $\overline{f}$ belongs to $\mathrm{Im}\, {}_{w_0}\varphi_{w_2}$. First notice that we can choose another representative f' of $\overline{f}$, such that f' appears naturally as a composition of g a (w_0, w_{m_1}, w_1)-vfoam and h a (w_1, w_{m_2}, w_2)-vfoam *i.e.* a small vertical slice of f' where w_{m_1} and w_{m_2} meet is equal to w_1 times a small interval, with w_1 an ε_1-web. Note that w_1 may not be in W^{ε_1} and actually it may be elliptic, so that we are not done. However, because of the construction of $\mathbb{K}^{\varepsilon_1}$ and of theorem 2.1.29, we know that in $\widetilde{K^{\varepsilon}}$, we can decompose id_{w_1} as a finite sum:

$$\mathrm{id}_{w_1} = \overline{\sum_{i \in I} a_i b_i}$$

where I is finite and, for every i in I, a_i is a $(w_1, \varepsilon_1, w_i)$-vfoam and b_i is a $(w_i, \varepsilon_1, w_1)$-vfoam and w_i is an element of W^{ε_1}. Therefore,

$$\overline{f} = \overline{f'} = \overline{\sum_{i \in I} g a_i b_i h} = {}_{w_0}\varphi_{w_2}\left(\sum_{i \in I} \overline{g a_i} \otimes \overline{b_i h}\right),$$

and hence $\overline{f}$ is in $\mathrm{Im}\, {}_{w_0}\varphi_{w_2}$. This shows that ${}_{w_0}\varphi_{w_2}$ is surjective.

Injectivity: Let $x = \sum_{i \in I} \overline{f_i} \otimes \overline{g_i}$ be an element of $\mathrm{Ker}\, {}_{w_0}\varphi_{w_2}$, where the f_i's and the g_i's are respectively (w_0, w_{m_1}, w_i)-vfoams and (w_i, w_{m_2}, w_2)-vfoams for w_i in W^{ε_i}. The idea is to push the non trivial part of f_i on the right hand side of the tensor product in order to see it as a part of an element of $\mathbb{K}^{\varepsilon_2}$ acting on a another vfoam h.

Note that f_i (or a (w_0, w_{m_1}, w_i)-vfoam isotopic to f_i) appears naturally as a composition of h a $(w_0, w_{m_1}, w_{m_1}w_0)$-vfoam isotopic (not relatively to the boundary, see figure 2.10) to

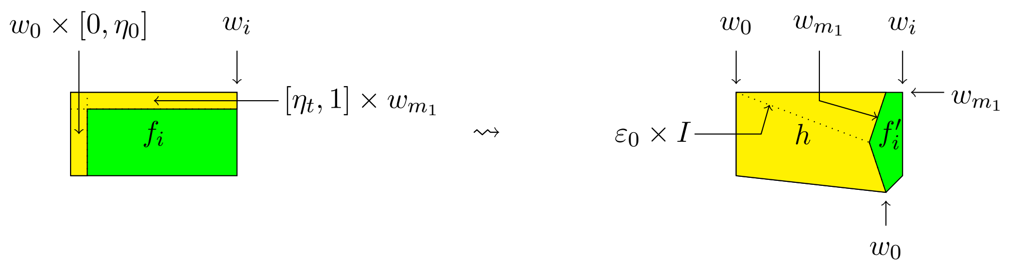


Figure 2.10: Each f_i decomposes as a composition of the $(w_0, w_{m_1}, w_{m_1}w_0)$-vfoam h and a $(w_0w_{m_1}, \varepsilon_1, w_i)$-vfoam.

$w_0w_{m_1} \times [0, 1]$, and f'_i a $(w_0w_{m_1}, \varepsilon_1, w_i)$-vfoam. Theorem 2.1.18 and proposition 2.2.8 tell us that the class $\overline{h}$ of $(w_0, w_{m_1}, w_{m_1}w_0)$-vfoam h can be expressed as a finite sum:

$$h = \sum_{j \in J} h_j a_j,$$

where for every j in J, h_j is a (w_0, w_{m_1}, w'_j)-vfoam and a_j belongs to ${}_{w'_j}(\mathbb{K}^{\varepsilon_1})_{w_{m_1}w_0}$ and w'_j is in W^{ε_1} for all j. Note that the foam h, the set J, the h_j's and the a_j's are independent from i. For all (i,j) in $I \times J$, $a_j f'_i$ belongs to ${}_{w_j}(\mathbb{K})_{w_i}$ so that in ${}_{w_0}\mathcal{F}(w_{m_1}) \otimes_{\mathbb{K}^{\varepsilon_1}} \mathcal{F}(w_{m_2})_{w_2}$ we have:

$$\overline{f_i \otimes g_i} = \overline{\sum_{j\in J} h_j a_j f'_i \otimes g_i} = \overline{\sum_{j\in J} h_j \otimes a_j f'_i g_i},$$

so that we can write:

$$\overline{\sum_{i\in I} f'_i \otimes g_i} = \overline{\sum_{i\in I}\sum_{j\in J} h_j \otimes a_j f'_i g_i} = \overline{\sum_{j\in J} h_j \otimes a_j \left(\sum_{i\in I} f'_i g_i\right)}.$$

Let us show that $\overline{\sum_{i\in I} f'_i g_i}$ is equal to zero. In fact, for all $i \in I$, as a $w_0 w_{m_1} w_{m_2} \overline{w_2}$-foam $f'_i g_i$ (forgetting the way we decomposed the border) is isotopic to $f_i g_i$, but we supposed that $\overline{\sum_{i\in I} f_i g_i}$ is equal to 0. This proves that $\overline{\sum_{i\in I} f'_i g_i}$ is equal to 0 and hence that ${}_{w_0}\varphi_{w_2}$ is injective.

We conclude that φ is an isomorphism. □

All the results and constructions of this section can be sum up in the following theorem:

Theorem 2.2.17 (Algebroid version)**.** *The functor $\mathcal{F}$ of theorem 1.2.10 extends into a 2-functor from the 2-category $\mathfrak{sl}_3$-2-Foam to the 2-category of finitely generated graded R-algebroids. The algebroid associated with a sequence of signs ε is $\mathbb{K}^\varepsilon$.*

The whole construction can be done using the algebras K^ε instead of the algebroids $\mathbb{K}^\varepsilon$ (see remark 2.2.9).

Theorem 2.2.17 (Algebra version)**.** *The functor $\mathcal{F}$ of theorem 1.2.10 extends into a 2-functor from the 2-category $\mathfrak{sl}_3$-2-Foam to the 2-category of finitemy generated graded R-algebras. The algebra associated with a sequence of signs ε is K^ε.*

2.2.4 The space of modules maps

In chapters 3 and 4, we will study the category of $\mathbb{K}^\varepsilon$-modules. For this purpose we will have a special look at the graded space of morphisms between $\mathbb{K}^\varepsilon$-modules. As we will see here, a lot of information is given by the Kuperberg bracket.

Notation 2.2.18. Let w be an ε-web, then we denote by P_w the $\mathbb{K}^\varepsilon$-module $\mathcal{F}(w)$, we call such a $\mathbb{K}^\varepsilon$-module a *web-module.*

Proposition 2.2.19. *As a $\mathbb{K}^\varepsilon$-module, $\mathbb{K}^\varepsilon$ is isomorphic to $\bigoplus_{w\in W^\varepsilon} P_w$, where W^ε is a set of representative of isotopy classes of non-elliptic ε-webs (see definition 2.2.7).*

Proof. Using its structure of module-$\mathbb{K}^\varepsilon$, we can write that $\mathbb{K}^\varepsilon$ as a direct sum of $\mathbb{K}^\varepsilon$-modules:

$$\mathbb{K}^\varepsilon = \bigoplus_{w'\in W^\varepsilon} (\mathbb{K}^\varepsilon)_{w'}.$$

We claim that P_w and $(\mathbb{K}^\varepsilon)_w$ are isomorphic as $\mathbb{K}^\varepsilon$-modules. The isomorphism is given on figure 2.11 at the level of foams (and vfoams). Note that as they both are $\mathbb{K}^\varepsilon$-modules, the, isomorphism splits as a direct sum of isomorphism between ${}_{w_0}(P_w)$ and ${}_{w_0}((\mathbb{K}^\varepsilon)_w)$ for w_0 running through W^ε. □

Proposition 2.2.20. *If w is an ε-web, then the $\mathbb{K}^\varepsilon$-module P_w is a projective module.*

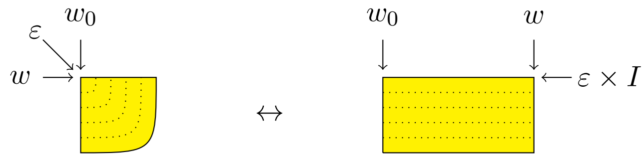


Figure 2.11: Isomorphism between ${}_{w_0}(P_w)$ (on the right) and ${}_{w_0}((\mathbb{K}^\varepsilon)_w)$ (on the left) explicited at the level of foams (and vfoams).

Proof. If w is non-elliptic, it's clear from proposition 2.2.19 that P_w is a direct factor of $\mathbb{K}^\varepsilon$ and hence is projective. If w is not non-elliptic, then P_w is isomorphic to a finite direct sum of (maybe degree shifted) P_{w_i} with w_i in W^ε, therefor, it is as well a projective module. □

Proposition 2.2.21. *Let w_1 and w_2 be two ε-webs, then the graded R-module* $\hom_{\mathbb{K}^\varepsilon}(P_{w_1}, P_{w_2})$ *is free and has dimension equal to $\langle(\overline{w_1}w_2)\rangle \cdot q^{l(\varepsilon)}$.*

Proof. An element of $\hom_{\mathbb{K}^\varepsilon}(P_{w_1}, P_{w_2})$ is completely determined by the image of 1_{w_1}. The image belongs to ${}_{w1}(P_{w_2})$. This shows that the $\mathbb{K}^\varepsilon$-module maps between P_{w_1} and P_{w_2} are all represented by (w_1, w_2)-foams. On the other hand, proposition 2.2.4 gives that the free R-module spanned by (w_1, w_2)-foams modded out by isotopy and by the foam relation **FR** is free and has graded dimension equal to $\langle(\overline{w_1}w_2)\rangle \cdot q^{l(\varepsilon)}$. We can conclude that:

$$\dim_q(\hom_{\mathbb{K}^\varepsilon}(P_{w_1}, P_{w_2})) = \langle(\overline{w_1}w_2)\rangle \cdot q^{l(\varepsilon)}.$$

□

In the following chapters, we will study the decomposability of web-modules. To show that a module is indecomposable, it is enough to show that its ring of endomorphisms contains no non-trivial idempotents. It appears that an idempotent must have degree zero, so we have the following lemma :

Lemma 2.2.22. *If w is a ε-web such that $\langle\overline{w}w\rangle$ is monic of degree $l(\varepsilon)$, then the graded $\mathbb{K}^\varepsilon$-module P_w is indecomposable.*

Proof. It follows from the previous discussion: if $\hom_{\mathbb{K}^\varepsilon}(P_w, P_w)$ contained a non-trivial idempotent, there would be at least two linearly independent elements of degree 0, but $\dim((\hom_{\mathbb{K}^\varepsilon}(P_w, P_w)_0) = a_{-l(\varepsilon)}$ where $\langle\overline{w}w\rangle = \sum_{i\in\mathbb{Z}} a_i q^i$. As $\langle\overline{w}w\rangle$ is symmetric (in q and q^{-1}) of degree $l(\varepsilon)$ and monic, $a_{-l(\varepsilon)}$ is equal to 1 and this is a contradiction. □

We have a similar lemma to prove that two modules are not isomorphic.

Lemma 2.2.23. *If w_1 and w_2 are two ε-webs such that $\langle\overline{w_1}w_2\rangle$ has degree strictly smaller than $l(\varepsilon)$, then the graded $\mathbb{K}^\varepsilon$-modules P_{w_1} and P_{w_2} are not isomorphic.*

Proof. If they were isomorphic, there would exist two non-trivial morphisms f and g such that $f \circ g = 1_{P_{w_1}}$ and therefore $f \circ g$ would have degree zero. The hypothesis made implies that f and g (because $\langle\overline{w_1}w_2\rangle = \langle\overline{w_2}w_1\rangle$) have positive degree so that the degree of their composition is as well positive. □

2.3 The $\mathfrak{sl}_3$-homology for tangles

The (0+1+1)-TQFT defined above allows us to construct a $\mathfrak{sl}_3$-homology for tangles. In this section we describe this homology. The construction is similar to the one of section 1.3 but in the tangle context.

We will consider oriented tangles with borders separated in two parts *i.e.* one part on the top and one part on the bottom, this fit into a category, objects are sequences of signs and morphisms from ε_0 to ε_1 are oriented tangles with border $-\varepsilon_0$ on the bottom and ε_1 on the top (we call these tangles oriented $(\varepsilon_0, \varepsilon_1)$*-tangles*). The way to compose morphisms is the usual way to compose tangles. We restrict to the case of admissible sequences of signs.

Definition 2.3.1. Let T be an oriented $(\varepsilon_0, \varepsilon_1)$-tangle diagram. *A smoothing function for* T is a function ϕ from the set of crossings of T to $\{0, 1\}$. Let c be a crossing of T and ϕ is a smoothing function for T such that $\phi(c) = 0$, we denote by ϕ_c the smoothing function equal to ϕ everywhere but on c and with $\phi_c(c) = 1$. The *size of* ϕ is the number of times it takes the value 1. It is denoted by $|\phi|$.

Definition 2.3.2. Let T be a $(\varepsilon_0, \varepsilon_1)$-tangle diagram, and ϕ a smoothing function for T. Then we define *the* ϕ*-smoothing of* T to be the $(\varepsilon_0, \varepsilon_1)$-web obtain from T by replacing each crossing c by its $\phi(c)$-smoothing (see figure 1.12 for definitions of 0-smoothing and 1-smoothing) and with a degree shift of $|\phi|$. We denote it by T_ϕ.

Definition 2.3.3. Let ε_0 and ε_1 be two admissible sequences of signs, T be a $(\varepsilon_1, \varepsilon_2)$-tangle diagram. Let I be the set of crossings of T. We define $H(T)$, *the hypercube of smoothings of* T, to be the I-hypercube where H_ϕ is the image of T_ϕ by the functor $\mathcal{F}$, and for ϕ a smoothing function and c a crossing such that $\phi(c) = 0$, the differential d^c_ϕ is the image by $\mathcal{F}$ of the foam which is everywhere identity but next to the crossing c and there it's given by figure 1.13. The *complex* $C(T)$ *of smoothings of* T is the complex $C(H(T))\{3n_- - 2n_+\}[-n_-]$. Note that, in this context the functor $\mathcal{F}$ gives us $\mathbb{K}^{\varepsilon_0}$-modules-$\mathbb{K}^{\varepsilon_1}$ and maps of $\mathbb{K}^{\varepsilon_0}$-modules-$\mathbb{K}^{\varepsilon_1}$ so that in the end the complex $C(H(T))$ is a complex of $\mathbb{K}^{\varepsilon_0}$-modules-$\mathbb{K}^{\varepsilon_1}$.

Theorem 2.3.4. *If* T_1 *is a* $(\varepsilon_0, \varepsilon_1)$*-tangle diagram and* T_2 *is a* $(\varepsilon_1, \varepsilon_2)$*-tangle diagram then we have an isomorphism of complexes of* $\mathbb{K}^{\varepsilon_0}$*-module-*$\mathbb{K}^{\varepsilon_2}$*:*

$$C(T_1T_2) \simeq C(T_1) \otimes_{\mathbb{K}^{\varepsilon}_1} C(T_2).$$

Proof. This is a direct application of 2.2.16. □

Theorem 2.3.5 (adapted from Khovanov [Kho04])**.** *If two* $(\varepsilon_0, \varepsilon_1)$*-tangle diagrams represent the same* $(\varepsilon_0, \varepsilon_1)$*-tangle, then, their complexes of smoothings are homotopic as complexes of* $\mathbb{K}^{\varepsilon_0}$*-module-*$\mathbb{K}^{\varepsilon_1}$*.*

Chapter 3

Superficial webs

The categories of modules over the algebras K^ε are more complicated than their analogues in the $\mathfrak{sl}_2$ case (see the definitions of the algebras H^n in [Kho02])). The reason of this fact is that the non-elliptic webs, which from a skein-module point of view, are irreducible can lead via the categorification to decomposable modules. In this chapter we first point out an example of this phenomenon, and then we give a rather large family of non-elliptic webs which behave well *i.e.* whose associated modules are indecomposable. This chapter is based on [Rob12]. We mention that modules over K^ε are studied independently in [MPT12] where they compute the split Grothendieck group of K^ε. All along this chapter the functor $\mathcal{F}$ we use is the one defined in chapter 2 with the ground ring R equal to $\mathbb{Q}$ (in particular, the algebras K^ε are $\mathbb{Q}$-algebras).

3.1 A decomposable module

As we discussed before, to prove that a module is decomposable, it's enough to show that its ring of endomorphisms contains a non-trivial idempotent. In this subsection we show that a certain module P_w is decomposable. This is actually already known (see for example [MN08]), but we give here details of the calculus.

In what follows, we set ε to be the sequence $(+,-,-,+,+,-,-,+,+,-,-,+)$ (so that $l(\varepsilon) = 12$). The ε-webs w and w_0 are given on figure 3.1. We will as well need some specific

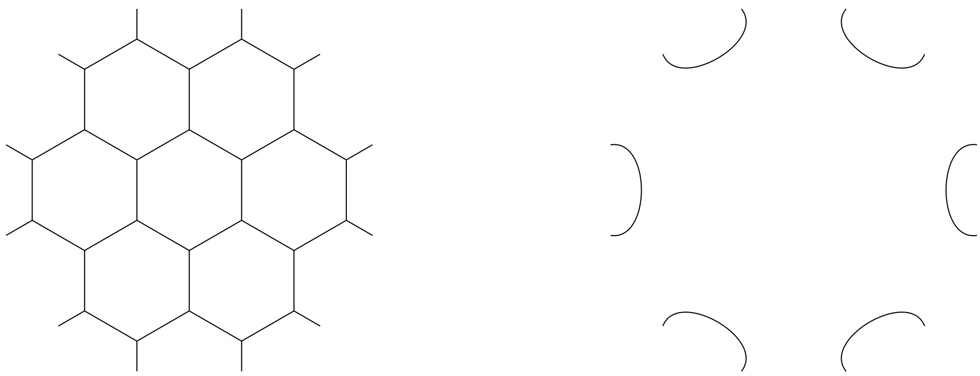


Figure 3.1: The ε-webs w (on the left) and w_0 (on the right), to fit in formal context of the 2-category one should stretch the outside edges to horizontal line below the whole picture, we draw it this way to enjoy more symmetry. To simplify we didn't draw the arrows.

foams. We describe them via movies on figure 3.2 (the elementary movies are birth, death and saddle as for classical surfaces and we add zip and unzip given on figure 1.13, see [Kho04] for details).

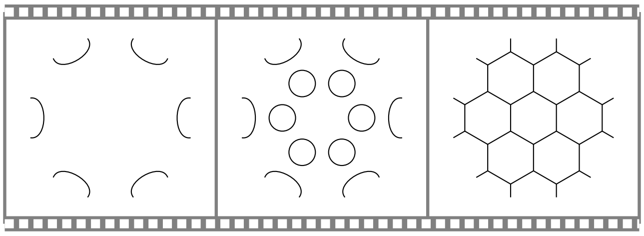


Figure 3.2: The (w_0, w)-foam f (it should be seen as an element of $\hom_{K^\varepsilon}(P_w, P_{w_0})$) is described by the movie from left to right and the (w, w_0)-foam g is described by the movie from right to left (we have $g = \overline{f}$). We specify for f : at the first step we perform 6 births and then, at the second step we zip 12 times, this leads to morphisms of degree 0.

Proposition 3.1.1. *The K^ε-module P_w is decomposable.*

First we should quote that this web was the counter-example pointed out by Kuperberg and Khovanov [KK99] to show that the web basis is not dual canonical. To show this proposition we just have to find a non-trivial idempotent.

Proof. We claim that $e \stackrel{\mathrm{def}}{=} \frac{1}{2} f \circ g$ is a (w, w)-foam whose associated endomorphism is an idempotent different from 0 and from 1_{P_w}. To prove this we first show that $g \circ f = 2 \cdot 1_{w_0}$. First notice that $g \circ f$ is a (w_0, w_0)-foam of degree 0 and belongs to $\hom_{K^\varepsilon}(P_{w_0}, P_{w_0})$. This space has a graded dimension given by $q^{12} \cdot \langle \overline{w_0} w_0 \rangle = q^{12} \cdot [3]^6$, and this shows that the space of degree 0 endomorphism of P_{w_0} is 1-dimensional and hence $g \circ f$ is a multiple of 1_{w_0}.

Seen as a foam and forgetting the decomposition of its boundary we can consider $g \circ f$ as $(\emptyset, \overline{w_0} w_0)$-foam and in this context $f \circ g$ is a multiple of h, the $(\emptyset, \overline{w_0} w_0)$-foam given by 6 cups. To evaluate the scalar multiple between them, we complete these $(\emptyset, \overline{w_0} w_0)$-foams to obtain closed foams by gluing j, the $(\overline{w_0} w_0, \emptyset)$-foam which consists of 6 caps with two dots on each cap. The foam $h \circ j$ is a closed foam which consists of 6 spheres with 2 dots on each, hence it's evaluation $\mathcal{F}(h \circ j)$ is equal to $(-1)^6 = 1$. Now let us evaluate $\mathcal{F}(g \circ f \circ j)$. Using the bubble relations next to the cups of j, we have $\mathcal{F}(g \circ f \circ j) = (-1)^3 \mathcal{F}(t)$, where t is a torus with 6 disks inside and one dot per section of the torus (see figure 3.3).

To evaluate $\mathcal{F}(t)$ one can perform surgeries on all portions, this gives us a priori 3^6 terms with plus signs, all are disjoint union of 6 dotted theta foams just 2 of this terms are non-zero: and for the two of them 3 theta foams evaluate on -1 and 3 theta foams evaluate on $+1$, so that $\mathcal{F}(g \circ f \circ j) = -\mathcal{F}(t) = 2$.

This gives us that $g \circ f = 2 \cdot 1_{w_0}$. It is then very easy to check that e is an idempotent: $e \circ e = \frac{1}{4} g \circ f \circ g \circ f = \frac{1}{2} g \circ 1_{w_0} \circ f = e$, and it is as well straightforward to check that e is not equal to zero: $f \circ e \circ g = 2 \cdot 1_{w_0} \neq 0$. We now need to show that e is not equal to 1_w. If it were so, f and $\frac{1}{2} g$ would be mutually inverse isomorphisms between P_w and P_{w_0}. But the spaces of endomorphisms of these two modules do not have the same graded dimensions[1] so they cannot be isomorphic. This shows that P_w is decomposable and that

[1] In fact we have $\langle w \overline{w} \rangle = 2q^{-12} + 80q^{-10} + 902q^{-8} + 4604q^{-6} + 13158q^{-4} + 23684q^{-2} + 28612 + 23684q^2 + 13158q^4 + 4604q^6 + 902q^8 + 80q^{10} + 2q^{12}$ and $\langle w_0 \overline{w_0} \rangle = q^{-12} + 6q^{-10} + 21q^{-8} + 50q^{-6} + 90q^{-4} + 126q^{-2} + 141 + 126q^2 + 90q^4 + 50q^6 + 21q^8 + 6q^{10} + q^{12}$. We used Lukas Lewark's program [Lew11] to compute $\langle w \overline{w} \rangle$.

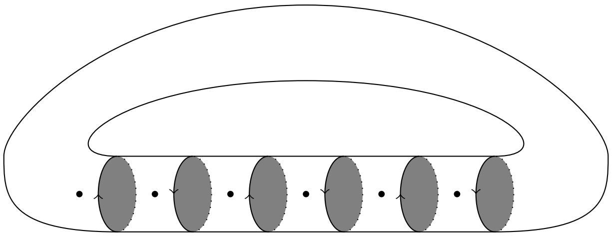


Figure 3.3: The closed foam t.

P_{w_0} is a direct factor of P_w. □

3.2 Superficial webs leads to indecomposable modules

The aim of this section is to give a rather large family of ε-webs whose associated K^ε-modules are indecomposable, and pairwise non-isomorphic.

3.2.1 Superficial webs, semi-non elliptic webs

Definition 3.2.1. If ε is an admissible sequence of signs, we denote by S^ε the $\mathbb{Q}[q, q^{-1}]$-module generated by isotopy classes of ε-webs and subjected to the Kuperberg relations (see proposition 1.1.12).

Proposition 3.2.2 (Kuperberg)**.** *The $\mathbb{Q}[q, q^{-1}]$-module S^ε is free and freely generated by the non-elliplitic ε-webs.*

Let us consider a ε-web, there are finitely many connected components of $\mathbb{R}^2 \setminus w$ (we call them *faces* even if some may not be homeomorphic to a disk). As w is compact, there is just one of these faces which is unbounded. We call it *the unbounded face.* Note that because of the geometric requirements on ε-webs, all the points of ε are in the adherence of the unbounded face. We say that two faces are *adjacent* if an edge of w is included in the intersection of their adherences.

Definition 3.2.3. A face of an ε-web is said to be *nested* if it is not adjacent to the unbounded face. An ε-web with no nested face is called *superficial.*

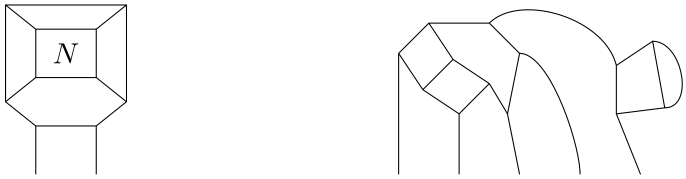


Figure 3.4: On the left, an (elliptic) ε-web with a nested face (marked by a N), on the right, a superficial (and elliptic) ε-web with two blocks.

The aim of this section is to prove the next theorem:

Theorem 3.2.4. *Let ε be an admissible sequence of signs and w be a superficial and non-elliptic ε-web, then the K^ε-module P_w, is indecomposable. Furthermore, if w' is another superficial and non-elliptic ε-web different*[2] *from w, then P_w and $P_{w'}$ are not isomorphic as K^ε-modules.*

[2]We mean non-isotopic.

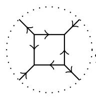
We begin by a few technical definitions. In an ε-web, let us consider all the faces but the unbounded one. They come in adjacency classes. We call such an adjacency a *block*. In other words, blocks are connected components of the graph obtained from the dual graph by removing the vertex corresponding to the unbounded component and all the edges involving this vertex.

Definition 3.2.5. An ε-web is *semi-non-elliptic* if it contains no circle, no digon, and at most one square per block.

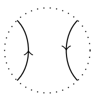
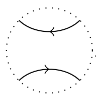
Lemma 3.2.6. *If an ε-web w is superficial and semi-non-elliptic then in the skein module S^{ε} it is equal to a sum of superficial and non-elliptic ε-webs with less vertices.*

Here, by "sum" here we mean linear combination with only positive integer coefficients.

Proof. We prove this by induction on the number of trivalent vertices. If w is already non-elliptic then there is nothing to prove. Else there is at least a square somewhere in w. Then if

$w =$  ,

we have

$w =$  $+$  .

Let w_1 and w_2 be these two webs. As w is superficial, one of the 4 faces around the square should be the unbounded face U, we can suppose it's the one on the top. We'll now inspect the faces of w_1 and w_2, see figure 3.5 for names of faces.

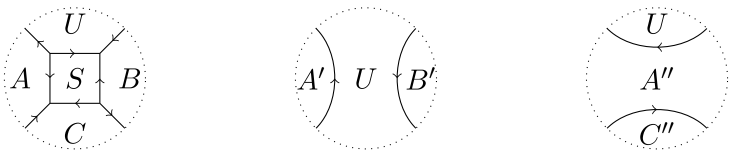


Figure 3.5: The webs w, w_1 and w_2.

The faces A' and B' of w_1 may be two squares but as w is superficial, A' and B' are on different blocks, else there would exist a path of faces in w from A to B disjoint of C and S and C would be nested. As the square S is the only square of its block in w, the block of A' and the block of B' have at most one square, hence w_1 is superficial and semi-non-elliptic. For w_2 now, the only possibly new square is the face C and if it is a square, it's the only one in its block, hence w_2 is superficial and semi-non-elliptic. To conclude, just notice that w_1 and w_2 have less vertices than w. $\square$

Definition 3.2.7. An ε-web w is called 1-elliptic if it contains no circle, no digon and if there are at most one square in each block except in one where it can contains at most two.

Lemma 3.2.8. *If w is a superficial 1-elliptic ε-web then there exist some non-elliptic ε-webs w_i and some symmetric polynomials in $\mathbb{N}[q, q^{-1}]$ with degree at most 1 such that in S^{ε} we have $w = \sum P_i w_i$.*

Proof. We prove this result by recursion on the number of vertices. If there are no vertices the web is non-elliptic and this is done. If w is semi-non-elliptic then the result comes from lemma 3.2.6. So it remains to understand the case where there is one block with two squares. If the two squares are far from each other (we mean that they don't share any edge) then we can proceed as in the proof of lemma 3.2.6 and prove that w is a sum (with positive integer coefficient) of 1-elliptic webs and then we conclude by recursion. Now we study the case where the two squares touch each other. As w is superficial the two squares S_1 and S_2 (see figure 3.6 for the notations) should touch the unbounded face. Furthermore w is equal to $[2]w_1 + w_2$ (see figure 3.7). Now let us consider the different cases. There are

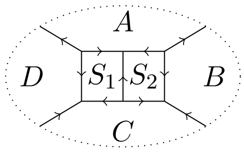


Figure 3.6: Situation where two squares touch each other.

two different situations, either A or C is unbounded (the two situations are symmetric) or both B and D are unbounded. If A is unbounded, then D' and B' may be squares but on different blocks, otherwise, C would be nested, and hence w_1 is superficial semi-non-elliptic. On the other hand B'' and D'' have at least 6 vertices, and C'' may be a square but in every case the ε-web w_2 is superficial semi-non-elliptic. It remains the case where B and D are unbounded. The face A' has at least 6 vertices so that w_1 is semi-non-elliptic. On the other hand A'' and C'' may be squares but there are clearly on different blocks so that w_2 is semi-non-elliptic. Hence using lemma 3.2.6 we conclude. $\square$

Figure 3.7: The webs w_1 and w_2.

Definition 3.2.9. An ε-web is said to be *semi-superficial* if it contains no circle, no digon and only one square and only one nested face and the nested face is an hexagon and the square and the nested hexagon share a side.

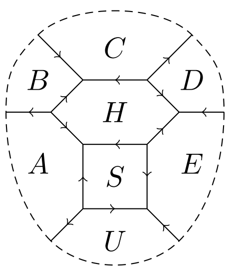


Figure 3.8: Semi-superficial web. The faces A, B, C, D, E, H and S are bounded. The label U shows the undounded face.

Lemma 3.2.10. *If w is a semi-superficial ε-web then there exist some non-elliptic ε-webs w_i and some symmetric polynomials in $\mathbb{N}[q, q^{-1}]$ with degree at most 1 such that in S^ε we have $w = \sum P_i w_i$.*

Proof. We take the notations of figure 3.8. We perform the square reduction on S and then on H in the configuration where it's possible so that in the skein module S^ε we have: $w = w_1 + w_2 + w_3$. See figure 3.9 for a description of the w_i.

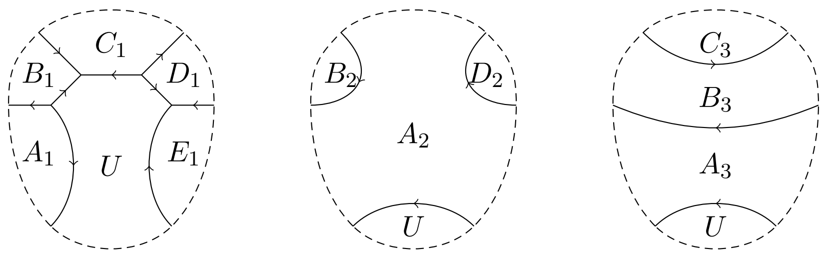


Figure 3.9: From left to right w_1, w_2 and w_3.

1. The web w_1 is superficial and 1-elliptic. The superficiality is clear, the only two faces which may be squares are A_1 and E_1.
2. The web w_2 is superficial and 1-elliptic. The face A_2 has at least 6 sides because it has the sides of A minus three of them and the sides of E minus three of them then it leads to at least 6 sides. The superficiality is clear, the only two faces which may be squares are B_2 and D_2.
3. The web w_3 is superficial and semi-non-elliptic. The superficiality is clear. The only face which can be a square is C_3.

From (1), (2) and (3), and the lemmas 3.2.8 and 3.2.6 we conclude easily. □

Thanks to lemma 2.2.22 and 2.2.23, to prove the theorem 3.2.4, it's enough to prove the following lemma:

Key lemma. *Let ε be a sequence of signs of length n. Let w_1 and w_2 be ε-webs, suppose w_1 and w_2 are superficial and non-elliptic. If $w_1 = w_2$ then $\langle \overline{w_1} w_2 \rangle$ is monic and has degree n, else* $\deg\langle \overline{w_1} w_2 \rangle < n$.

3.2.2 Proof of the key lemma

Definition 3.2.11. We consider the set W of (isotopy class) of pairs of superficial and non-elliptic webs with the same boundary. On this set we define a partial order: $(w_1, w_2) < (w_1', w_2')$ if and only if either $l(\partial w_1) < l(\partial w_1')$ or $l(\partial w_1) = (\partial w_1')$ and $\#V(w_1) + \#V(w_2) < \#V(w_1') + \#V(w_2')$. An element $\mathbf{w} = (w_1, w_2)$ of W is *symmetric* if $w_1 = w_2$.

This order is meant to encode the complexity of a web. We could have sharpened it but this won't be necessary for our purposes. For $n \in \mathbb{N}$, we denote W_n the subset of W in which the webs in the pairs have a boundary of length n. For example $W_0 = \{(\emptyset, \emptyset)\}$ and $W_1 = \emptyset$. As every W_n is finite, W is a well quasi-ordered set with one minimal element: $(\emptyset, \emptyset)$.

Definition 3.2.12. Let $\mathbf{w} = (w_1, w_2)$ be an element of W_n, we say that $\mathbf{w}$ is *nice* if

- the element $\mathbf{w}$ is symmetric, $\langle \overline{w_1} w_2 \rangle$ is monic and has degree n,
- or if $\mathbf{w}$ is not symmetric and the degree of $\langle \overline{w_1} w_2 \rangle$ is strictly smaller than n.

The lemma is rephrased in this vocabulary by the following proposition:

Proposition 3.2.13. *Every element of W is nice.*

Notation 3.2.14. If $\mathbf{w} = (w_1, w_2)$ is an element of W, then $\overline{w_1}w_2$ is a closed web, in what follows it will be practical to consider the isotopy class of $\overline{w_1}w_2$ but to keep in mind that this closed web has two different parts: the w_1 part and the w_2 part. In order to remember this, while performing an isotopy of $\overline{w_1}w_2$ we keep track of the real line where ε was lying (this is where $\overline{w_1}$ and w_2 are glued together). This curve is the *border* between w_1 and w_2. This will be depicted by a dashed line.

When performing a reduction of $\overline{w_1}w_2$ (removing a circle or a reduction of a square or of a digon), we should keep the border in mind so that the reduction leads to a new closed web w' which can be understood as $\overline{w'_1}w'_2$, where w'_i is the same as w_i except in the place where we perform the reduction. Note that when the reduction takes place next to the border we have to specify how the border behaves with respect to the reduction so that w'_1 and w'_2 are well defined. We may as well perform moves of the border, this is to be understood as that we change the pair (w_1, w_2) in the way given by the changed border (this may change the boundary).

Lemma 3.2.15. *If $\mathbf{w} = (w_1, w_2)$ is an element of W such that $\overline{w_1}w_2$ contains a circle C then there exists $\mathbf{w}' = (w'_1, w'_2)$ with $\mathbf{w}' < \mathbf{w}$ such that $\mathbf{w}'$ nice implies $\mathbf{w}$ nice.*

Proof. As w_1 and w_2 are non-elliptic, the circle must intersect the border. We shall consider two cases: the border cuts C in two points or in at least four points[3].

First consider the case where the border cuts C in two points. Then it separates the C into two half-circles $C_i \subset w_i$ for $i \in \{1, 2\}$. We denote w'_i the ε'-web $w_i \setminus C_i$. The sequence ε' is equal to ε with a '+' and a '−' removed hence $l(\varepsilon') = l(\varepsilon) - 2$. We have $\langle \overline{w_1}w_2 \rangle = [3] \left\langle \overline{w'_1}w'_2 \right\rangle$ and $\mathbf{w}'$ is symmetric if and only if $\mathbf{w}$ is. It's clear that if $\mathbf{w}' = (w'_1, w'_2)$ belongs to W and $\mathbf{w}' < \mathbf{w}$ and that if $\mathbf{w}'$ is nice then $\mathbf{w}$ is nice.

The second case: C meets the border in a least four points. Then $w_1 \neq w_2$. Consider once more the pair of ε'-webs $\mathbf{w}' = (w'_1, w'_2)$ obtained from $\mathbf{w}$ by removing the circle C. The length of ε' is at most $l(\varepsilon) - 4$. Then if $\mathbf{w}'$ is nice then $\deg \left\langle \overline{w'_1}w'_2 \right\rangle \leqslant l(\varepsilon) - 4$, and then:

$$\deg \langle \overline{w_1}w_2 \rangle = \deg \left([3] \left\langle \overline{w'_1}w'_2 \right\rangle \right) = 2 + \deg \left\langle \overline{w'_1}w'_2 \right\rangle \leqslant l(\varepsilon) - 2.$$

This is clear that $\mathbf{w}'$ is in W and that $\mathbf{w}' < \mathbf{w}$. And so we are done. □

Lemma 3.2.16. *If $\mathbf{w} = (w_1, w_2)$ is an element of W such that $\overline{w_1}w_2$ is not connected then there exist $\mathbf{w}'$ and $\mathbf{w}''$ with $\mathbf{w}' < \mathbf{w}$ and $\mathbf{w}'' < \mathbf{w}$ such that if $\mathbf{w}'$ and $\mathbf{w}''$ are nice then $\mathbf{w}$ is nice.*

Proof. Consider one connected component of $\overline{w_1}w_2$ and denote it by u and denote v the complement of u in $\overline{w_1}w_2$. Denote w'_1 and (resp. w'_2) the sub-web of w_1 (resp. of w_2) such that $\overline{w'_1}w'_2 = u$ and let w''_1 (resp. w''_2) be the complementary web of w'_1 in w_1. (resp. of w'_2 in w_2). Denote ε' the boundary of w'_1 and ε'' the boundary of w''_1. Let $\mathbf{w}'$ be (w'_1, w'_2) and $\mathbf{w}''$ be (w''_1, w''_2), it's clear that $\mathbf{w}$ and $\mathbf{w}''$ belong to W. We have $l(\varepsilon') + l(\varepsilon) = l(\varepsilon)$, so that $\mathbf{w}'$ and $\mathbf{w}''$ are smaller than $\mathbf{w}$. It's clear that $\mathbf{w}$ is symmetric if and only if $\mathbf{w}'$ and $\mathbf{w}''$ are. We have

$$\langle \overline{w_1}w_2 \rangle = \left\langle \overline{w'_1}w'_2 \right\rangle \cdot \left\langle \overline{w''_1}w''_2 \right\rangle,$$

so that if $\mathbf{w}'$ and $\mathbf{w}''$ are nice, then $\mathbf{w}$ is nice. □

[3]A circle and the border must intersect in an even number of points for orientation reasons.

Lemma 3.2.17. *Let $\mathbf{w} = (w_1, w_2)$ be an element of W such that $\overline{w_1}w_2$ is connected and contains a digon B which intersects the border of $\mathbf{w}$ in exactly one point per side of the digon. Then there exists $\mathbf{w}' = (w_1', w_2')$ with $\mathbf{w}' < \mathbf{w}$ such that $\mathbf{w}'$ nice implies $\mathbf{w}$ nice.*

Figure 3.10: Case of the digon with one vertex by side: on the left $\overline{w_1}w_2$, on the right $\overline{w_1'}w_2'$.

Proof. The situation is illustrated on figure 3.10. We perform the digon reduction by deleting one edge of the digon, reversing the orientation of the other one and forgot the two 2-valent vertices. We obtain a new pair of ε'-webs $\mathbf{w}' = (w_1', w_2')$. This is an element of W, in fact this is clear that the two webs are superficial ; to see that they are non-elliptic, one can notice that from w_i to w_i' we removed just one vertex and this vertex is not adjacent to any face but the unbounded one, and consequently the non-ellipticity is preserved. It's clear that $\mathbf{w}$ is symmetric if and only if $\mathbf{w}'$ is, the length of ε' is equal to $l(\varepsilon) - 1$ so $\mathbf{w}' < \mathbf{w}$ and we have $\langle \overline{w_1}w_2 \rangle = [2] \left\langle \overline{w_1'}w_2' \right\rangle$, so if $\mathbf{w}'$ is nice then $\mathbf{w}$ is nice. $\square$

Lemma 3.2.18. *Let $\mathbf{w} = (w_1, w_2)$ be an element of W such that $\overline{w_1}w_2$ contains a digon B which intersects the border of $\mathbf{w}$ in exactly two points which are on the same side. Then there exists a finite collection $\left(\mathbf{w}^{(i)}\right)$ with $\mathbf{w}^{(i)} < \mathbf{w}$ for all i, such that $\mathbf{w}^{(i)}$ nice for all i implies $\mathbf{w}$ nice.*

Figure 3.11: Case of the digon with the two vertices on the same side: on the left $\overline{w_1}w_2$, on the right $\overline{w_1'}w_2'$.

Proof. The situation is illustrated by figure 3.11. In this case $\mathbf{w}$ cannot be symmetric. Without loss of generality[4] we can suppose that the two vertices of B are in w_1. We reduce the digon B as follows: delete the side meeting the border, reverse the orientation of the other edge, and forget the two 2-valent vertices denote $\mathbf{w}'$ the new pair of ε'-webs corresponding to the situation. The length of ε' is equal to $l(\varepsilon) - 2$. The ε'-web w_1' is clearly superficial and semi-non-elliptic so we can apply lemma 3.2.6 and we have a finite collection $w_1^{(i)}$ of superfical non-elliptic ε'-webs such that in the skein module $S^{\varepsilon'}$ we have $w_1' = \sum_i \lambda w_i^{(i)}$ for some positive integer λ_i. On the other hand w_2' is clearly superfical and non-elliptic. Denote $\mathbf{w}^{(i)} = (w_i^{(i)}, w_2')$. Suppose that all the $\mathbf{w}^{(i)}$ are nice. We have:

$$\deg \langle \overline{w_1}w_2 \rangle = \deg \left([2] \left\langle \overline{w_1'}w_2' \right\rangle\right) = 1 + \max_i \deg \left\langle \overline{w_1^{(i)}}w_2' \right\rangle \leqslant 1 + l(\varepsilon') = l(\varepsilon) - 1$$

And this shows that $\mathbf{w}$ is nice. $\square$

[4]The two vertices are on the same side since one of the edges joining them does not meet the border.

Proposition 3.2.19. *Let $\mathbf{w} = (w_1, w_2)$ be an element of W such that $\overline{w_1}w_2$ is connected and contains a digon B. Then there exists a finite collection $\left(\mathbf{w}^{(i)}\right)$ with $\mathbf{w}^{(i)} < \mathbf{w}$ for all i, such that $\mathbf{w}^{(i)}$ nice for all i implies $\mathbf{w}$ nice.*

Proof. If we are in the situation of 3.2.17 and 3.2.18 this is already done, so suppose that the border meets the digon B at least 3 times. The situation is illustrated on figure 3.12. In this case $\mathbf{w}$ cannot be symmetric. We denote D the disk delimited by the digon. As the web $\overline{w_1}w_2$ is connected we can suppose that the interior of D is disjoint from the web. Consider now the restriction of the border to D. It's the reunion of different arcs. Push one outer arc a outside D. This leads to a new pair of ε'-webs $\mathbf{w}' = (w_1', w_2')$. The length of ε' is equal to $l(\varepsilon) - 1$ (when the two extremities of a lie on two different sides of B) or to $l(\varepsilon) - 2$ (when the two extremities of B lie on the same side of B). The web $\mathbf{w}'$ is in W: The operation that we described does not disturb the non-ellipticity condition neither the superficiality condition. Furthermore $\mathbf{w}' < \mathbf{w}$, and if $\mathbf{w}'$ is nice $\deg \langle \overline{w_1}w_2 \rangle \leqslant l(\varepsilon) - 1$ so that $\mathbf{w}$ is nice. □

Figure 3.12: Remaining cases for the digon: we move the boundary. On the left the outer arc meets the two edges, on the right it meets two times the same edge. In both cases only the outer arc of the boundary is drawn, but the boundary meets the digon elsewhere as well.

Lemma 3.2.20. *Let $\mathbf{w} = (w_1, w_2)$ be an element of W such that $\overline{w_1}w_2$ is connected and contains a square S such that S intersects the border of $\mathbf{w}$ in two points on opposite sides. Then there exists a finite collection $\left(\mathbf{w}^{(k)}\right)$ with $\mathbf{w}^{(k)} < \mathbf{w}$ for all k, such that if all the $\mathbf{w}^{(k)}$ are nice then $\mathbf{w}$ is nice.*

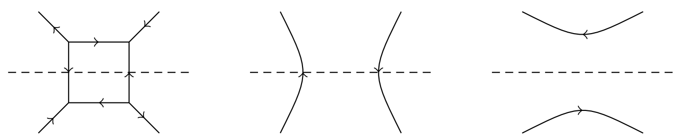


Figure 3.13: The border meets two opposite sides of the square: from left to right: $\overline{w_1}w_2$, $\overline{w_1'}w_2'$ and $\overline{w_1''}w_2''$. It's clear that w_1' and w_2' are superficial and non-elliptic, and that w_1'' and w_2'' are superfical semi-non-ellipitic. Furthermore, $w_1 = w_2$ if and only if $w_1' = w_2'$.

Proof. The situation is illustrated by figure 3.13. The connectedness hypothesis tells us that we can suppose the interior of the disk D delineated by S to be disjoint from the web. In D, the border is just a simple arc joining two opposite sides. We perform the two reductions of the square by deleting two opposite sides, reversing the orientations on the two lasting sides and forgetting the four 2-valent vertices. We obtain one pair of ε-webs $\mathbf{w}' = (w_1', w_2')$ (when keeping the sides which meet the border) and one pair of ε''-webs $\mathbf{w}'' = (w_1'', w_2'')$ with $l(\varepsilon'') = l(\varepsilon) - 2$ (when deleting the sides which meet the border). The

ε-webs w'_1 and w'_2 are superficial and non-elliptic so that $\mathbf{w}' \in W$. Because of the number of vertices we have $\mathbf{w}' < \mathbf{w}$. The ε''-webs w''_1 and w''_2 are superficial and semi-non-elliptic. Then there exists a finite collection $\left(w_1^{(i)}\right)$ (resp. $\left(w_2^{(j)}\right)$) of superficial and non-elliptic ε''-webs and some positive integers λ_i (resp. μ_j) such that in the skein module $S^{\varepsilon''}$ we have $w''_1 = \sum_i \lambda_i w_1^{(i)}$ and $w''_2 = \sum_j \mu_j w_2^{(j)}$. We have then:

$$\langle \overline{w_1} w_2 \rangle = \left\langle \overline{w'_1} w'_2 \right\rangle + \sum_{i,j} \lambda_i \mu_j \left\langle \overline{w_1^{(i)}} w_2^{(j)} \right\rangle .$$

We denote $\mathbf{w}^{(i,j)} = (w_1^{(i)}, w_2^{(j)})$. Suppose that $\mathbf{w}'$ is nice and that all the $\mathbf{w}^{(i,j)}$ are nice. It's straightforward that $\mathbf{w}$ is symmetric if and only if $\mathbf{w}'$ is, so that it's clear that, $\mathbf{w}$ is nice. □

Lemma 3.2.21. *Let $\mathbf{w} = (w_1, w_2)$ be an element of W such that $\overline{w_1} w_2$ is connected and contains a square S such that S intersects the border of $\mathbf{w}$ in two points on adjacent sides. Then there exists a finite collection $\left(\mathbf{w}^{(k)}\right)$ with $\mathbf{w}^{(k)} < \mathbf{w}$ for all k, such that if all the $\mathbf{w}^{(k)}$ are nice then $\mathbf{w}$ is nice.*

Figure 3.14: The boundary meets the square at two adjacent sides: on the left $\overline{w_1} w_2$, on the right $\overline{w'_1} w'_2$. It's clear that w'_2 is superficial and non-elliptic, and that w'_2 is superficial semi-non-ellipitic.

Proof. The situation is illustrated by figure 3.14. First notice that in this situation $\mathbf{w}$ cannot be symmetric. As before, the connectedness hypothesis allows us to suppose that the interior of the disk D delineated by S is disjoint from the web. The restriction of the border to D is just a simple arc connecting two adjacent sides. We move a little this arc: we push it outside the square through the common vertex of the two adjacent sides and we obtain a new pair of ε'-webs $\mathbf{w}' = (w'_1, w'_2)$ with $l(\varepsilon') = l(\varepsilon) - 1$. Now the square S doesn't meet the border anymore. With no loss of generality we can assume that it lies in w'_1. The ε'-web w'_2 is superficial and non-elliptic. The ε'-web w'_1 is superficial (the only faces which could be nested in w'_1 are the one next to the square, but they are obviously not) and semi-non-elliptic. So it exists a finite collection $\left(w_1^{(k)}\right)$ of non-elliptic superficial ε'-webs and some positive integers λ_k such that in the skein module $S^{\varepsilon'}$, $w_1 = \sum_k \lambda_k w_1^{(k)}$. Let $\mathbf{w}^{(k)} = (w_1^{(k)}, w'_2)$. It's clear that $\mathbf{w}^{(k)} < \mathbf{w}$ for all k. Suppose that all the $\mathbf{w}^{(k)}$ are nice.

$$\deg \langle \overline{w_1} w_2 \rangle = \deg \left(\sum_k \left\langle \overline{w_1^{(k)}} w_2 \right\rangle \right) \leqslant l(\varepsilon') = l(\varepsilon) - 1.$$

This proves that $\mathbf{w}$ is nice. □

We will now inspect the case where the border meets just one side of the square. This is the most technical part, so we need to separate it in different sub-cases. For this we will need to consider the face adjacent to the square and to the side opposite to the one that meets the border. We call this face the *opposed face*.

Lemma 3.2.22. *Let $\mathbf{w} = (w_1, w_2)$ be an element of W such that $\overline{w_1}w_2$ is connected and contains a square S. Suppose S intersects the border of $\mathbf{w}$ in exactly two points on the same side and the opposed face F or one of its neighbors different from the square meets the boundary. Then there exists a finite collection $\left(\mathbf{w}^{(k)}\right)$ with $\mathbf{w}^{(k)} < \mathbf{w}$ for all k, such that if $\mathbf{w}^{(k)}$ is nice for all k then $\mathbf{w}$ is nice.*

Figure 3.15: Illustration of lemma 3.2.22: on the left $\overline{w_1}w_2$, on the right $\overline{w_1'}w_2'$. It's clear that w_2' is superficial and non-elliptic, and that w_2' is superfical semi-non-ellipitic.

Proof. First notice that in this case $\mathbf{w}$ cannot be symmetric. The four vertices of S lie on the same side of the border, we can suppose that this is on the w_1 side. As usual, we can suppose that the interior of the disk D delineated by the square is disjoint from the web. The restriction of the border is just a simple arc in D joining one side to itself. We move the border locally by pushing it away from S in w_2. We obtain a new pair of ε'-webs $\mathbf{w}' = (w_1', w_2')$, with $l(\varepsilon') = l(\varepsilon) - 2$. It's clear that w_1' is superficial and semi-non-elliptic. So it exists a finite collection $\left(w_1^{(k)}\right)$ of non-elliptic superficial ε'-webs and some positive integers λ_k such that in the skein module $S^{\varepsilon'}$, $w_1 = \sum_k \lambda_k w_1^{(k)}$. Set $\mathbf{w}^{(k)} = (w_1^{(k)}, w_2')$. Suppose that all the $\mathbf{w}^{(k)}$ are nice. Then $\deg \langle \overline{w_1}w_2 \rangle \leqslant l(\varepsilon') - 2 < l(\varepsilon)$ and hence $\mathbf{w}$ is nice. □

Lemma 3.2.23. *Let $\mathbf{w} = (w_1, w_2)$ be an element of W such that $\overline{w_1}w_2$ is connected, contains no digon and contains a square S. Suppose S intersects the border of $\mathbf{w}$ in exactly two points on the same side and the opposed F face has at least 8 sides. Then there exists a finite collection $\left(\mathbf{w}^{(k)}\right)$ with $\mathbf{w}^{(k)} < \mathbf{w}$ for all k, such that $\mathbf{w}^{(k)}$ nice for all k implies $\mathbf{w}$ nice.*

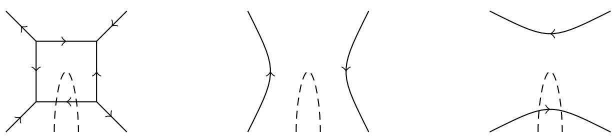


Figure 3.16: Illustration of lemma 3.2.23: from left to right: $\overline{w_1}w_2$, $\overline{w_1'}w_2'$ and $\overline{w_1''}w_2''$.

Proof. The situation is illustrated on figure 3.16 First notice that in this case $\mathbf{w}$ cannot be symmetric. The four vertices of S lie on the same side of the border, we can suppose that this is on the w_1 side. As usual, we can suppose that the interior of the disk D delineated by the square is disjoint from the web. The restriction of the border is just a simple arc in D joining one side to itself. We perform the two reductions of the square by deleting two opposite sides, reversing the orientations on the two lasting sides and forgetting the four 2-valent vertices. We obtain one pair of ε'-webs $\mathbf{w}' = (w_1', w_2')$ with $l(\varepsilon') = l(\varepsilon) - 2$ (when deleting the sides which meet the border) and one pair of ε-webs $\mathbf{w}'' = (w_1'', w_2'')$

(when keeping the sides which meet the border). The ε'-webs w_1' and w_2' are superficial and 1-elliptic. Thanks to lemma 3.2.8 there exist a finite collection $\left(w_1^{(i)}\right)$ of non-elliptic and superficial ε'-webs and a finite collection (P_i) of symmetric Laurent polynomial in $\mathbb{N}[q, q^{-1}]$ with degree at most 1 such that in the skein module $S^{\varepsilon'}$, $w_1' = \sum_i P_i w_1^{(i)}$. On the other hand, w_2' is superficial and non-elliptic. Denote $\mathbf{w}^{(i)} = (w_1^{(i)}, w_2')$. Let us inspect $\mathbf{w}''$ now, it's clear that the ε-web w_1'' is superficial and the hypothesis made on F implies that it is non-elliptic. The ε-web w_2'' is clearly non-elliptic and superficial. The hypothesis on the absence of digon implies that $\mathbf{w}''$ is not symmetric. Suppose that $\mathbf{w}''$ and all the $\mathbf{w}^{(k)}$ are nice. Then we have:

$$\begin{aligned}\deg \langle \overline{w_1} w_2 \rangle &= \deg \left(\left\langle \overline{w_1'} w_2' \right\rangle + \left\langle \overline{w_1''} w_2'' \right\rangle \right) = \deg \left(\sum P_i \left\langle \overline{w_1^{(i)}} w_2' \right\rangle + \left\langle \overline{w_1''} w_2'' \right\rangle \right) \\ &\leqslant \max(1 + l(\varepsilon'), l(\varepsilon) - 1) = l(\varepsilon) - 1.\end{aligned}$$

This proves that $\mathbf{w}$ is nice. □

Lemma 3.2.24. *Let $\mathbf{w} = (w_1, w_2)$ be an element of W such that $\overline{w_1} w_2$ is connected, contains no digon and contains a square S. Suppose S intersects the border of $\mathbf{w}$ in exactly two points on the same side and the opposed face F is an hexagon and does not meet the border. Then there exists a finite collection $\left(\mathbf{w}^{(k)}\right)$ with $\mathbf{w}^{(k)} < \mathbf{w}$ for all k, such that $\mathbf{w}^{(k)}$ nice for all k implies $\mathbf{w}$ nice.*

Proof. First notice that in this case $\mathbf{w}$ cannot be symmetric. The four vertices of S lie on the same side of the border, we can suppose that this is on the w_1 side. As usual, we can suppose that the interior of the disk D_1 delineated by the S and interior of the disk D_2 delineate by the hexagon are disjoint from the web. The restriction of the border is just a simple arc in D_1 joining one side to itself. We move the border by pushing the arc out of D_1 (this is the same move as in figure 3.15). We denote $\mathbf{w}' = (w_1', w_2')$ the new pair of ε'-webs with $l(\varepsilon') = l(\varepsilon) - 2$. The ε'-web w_1' is clearly semi-superficial and the ε'-web w_2' is superficial and non-elliptic. The lemma 3.2.10 tells us that there exists a finite collection $\left(w_1^{(i)}\right)$ of non-elliptic superficial ε'-webs and a finite collection (P_i) of symmetric Laurent polynomial in $\mathbb{N}[q, q^{-1}]$ with degree at most 1 such that in the skein module $S^{\varepsilon'}$, $w_1' = \sum_i P_i w_1^{(i)}$. Denote $\mathbf{w}^{(i)} = (w_1^{(i)}, w_2')$, and suppose that all the $\mathbf{w}^{(i)}$ are nice. Then

$$\deg \langle \overline{w_1} w_2 \rangle = \deg \left(\sum P_i \left\langle \overline{w_1^{(i)}} w_2' \right\rangle \right) \leqslant 1 + l(\varepsilon') = l(\varepsilon) - 1$$

This proves that $\mathbf{w}$ is nice. □

Proposition 3.2.25. *Let $\mathbf{w} = (w_1, w_2)$ be an element of W such that $\overline{w_1} w_2$ is connected, contains no digon and contains a square S. Then there exists a finite collection $\left(\mathbf{w}^{(k)}\right)$ with $\mathbf{w}^{(k)} < \mathbf{w}$ for all k, such that $\mathbf{w}^{(k)}$ nice for all k implies $\mathbf{w}$ nice.*

Proof. The border must cut the square S. prove do this by induction on the number of intersection points of the border with S. If the border meets S two times then the lemmas 3.2.20, 3.2.21, 3.2.22, 3.2.23 and 3.2.24 give the result. If it has more that two intersections points, then $\mathbf{w}$ is not symmetric. We move apart of the border (an outer arc) outside S without increasing the length of ε (in case the arc meets adjacent sides we do like in 3.2.21, in case it meets the same side two times we do the move described on figure 3.16, in case all the outer arcs meet two opposite sides we perform the move depicted on figure 3.17). The only thing to realize is that when one move an arc joining to opposite side of S out of S this cannot result to a symmetric $\mathbf{w}$ because of the no-digon hypothesis. These moves decrease the number of intersecting points of S with the border and we can use the recursion hypothesis. □

Figure 3.17: When all the outer arc meet the two sides, we move one of this arc out of the square.

Proof of theorem 3.2.4. As we said, this is enough to prove 3.2.13. We do this by induction with respect to the order on W. It's clear that the result is true for $(\emptyset, \emptyset)$. Suppose we have an element $\mathbf{w} = (w_1, w_2)$ of W such that for all $\mathbf{w}'$ of W with $\mathbf{w}' < \mathbf{w}$ then $\mathbf{w}'$ is nice. Then depending on how $\overline{w_1}w_2$ looks like we can apply lemma 3.2.16, lemma 3.2.16, proposition 3.2.19 or proposition 3.2.25, and this shows that $\mathbf{w}$ is nice. □

Chapter 4

A characterisation of indecomposable web-modules

The chapter 3 gives a sufficient condition for a web-module to be indecomposable. All the argumentation relies on the computation of the dimension of the space of the degree 0 endomorphisms of web-modules: in fact, when for a web w, this space has dimension 1, then the web-module P_w is indecomposable. Translated in terms of Kuperberg bracket, it says (see as well lemma 2.2.22):

> *If w is an ε-web such that $\langle \overline{w}w \rangle$ is monic of degree $l(\varepsilon)$, then the K^ε-module P_w is indecomposable.*

The aim of this chapter is to prove the converse. This will give the following characterisation of indecomposable web-modules:

Theorem. *Let w be an ε-web. The K^ε-module P_w is indecomposable if and only if $\langle \overline{w}w \rangle$ is monic of degree $l(\varepsilon)$. Furthermore if the K^ε-module P_w is decomposable it contains another web-module as a direct factor.*

The proof relies on some combinatorial tools called red graphs. In a first part we give an explicit construction (in terms of foams) of a non-trivial idempotent associated to a red graph. In a second part we show that when an ε-web w is such that $\langle \overline{w}w \rangle$ is not monic of degree $l(\varepsilon)$, then it contains a red graph.

4.1 Red graphs

4.1.1 Definitions

The red graphs are sub-graphs of the dual graphs webs, we recall here the definition of a dual graph. For an introduction to graph theory we refer to [Har69] and [BM08].

Definition 4.1.1. Let G be a plane graph (with possibly some vertex-less loops), we define *the dual graph $D(G)$ of G* to be the abstract graph given as follows:

- The set of vertices $V(D(G))$ of $D(G)$ is in one-one correspondence with the set of connected components of $\mathbb{R}^2 \setminus G$ (including the unbounded connected component). Such connected component are called *faces*.

- The set of edges of $D(G)$ is in one-one correspondence with the set of edges of G (in this construction, vertex-less loops are not seen as edges). If an edge e of G is adjacent to the faces f and g (note that f may be equal to g if e is a bridge), then the corresponding edge e' in $D(G)$ joins f' and g', the vertices of $D(G)$ corresponding to f and g.

Note that in general the faces need not to be diffeomorphic to disks. It is easy to see that the dual graph of a plane graph is planar: we place one vertex inside each face, and we draw an edge e' corresponding to e so that it crosses e exactly once and it crosses no other edges of G. Such an embedding of $D(G)$ is a *plane dual* of the graph G (see figure 4.1).

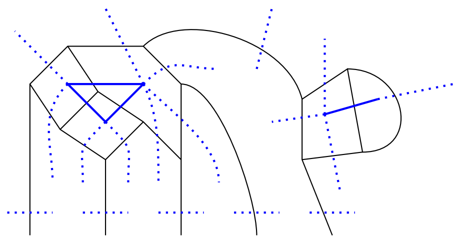


Figure 4.1: In black an ε-web w and in blue the dual graph of w. The dotted edges are all meant to belong to $D(w)$ and to reach the vertex u corresponding to the unbounded component of $\mathbb{R}^2 \setminus w$.

Definition 4.1.2. Let w be an ε-web, a *red graph* for w is a non-empty subgraph G of $D(w)$ such that:

(i) All faces belonging to $V(G)$ are diffeomorphic to disks. In particular, the unbounded face is not in $V(G)$.

(ii) If f_1, f_2 and f_3 are three faces of w which share together a vertex, then at least one of the three does not belong to $V(G)$.

(iii) If f_1 and f_2 belongs to $V(G)$ then every edge of $D(w)$ between f_1 and f_2 belongs to $E(G)$, *i.e.* G is an induced subgraph of $D(w)$.

If f is a vertex of G we define $\mathrm{ed}(f)$, *the external degree of* f, by the formula:

$$\mathrm{ed}(f) = \deg_{D(w)}(f) - 2\deg_G(f).$$

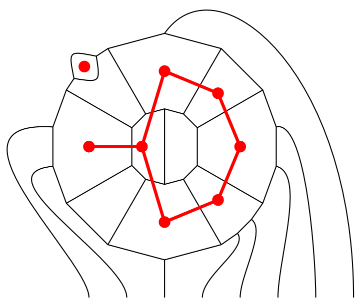


Figure 4.2: Example of a red graph.

Remark 4.1.3. Note that the external degree of a face f is always an even number because w being bipartite, all cycles are of even length and hence $\deg_{D(w)}$ is even.

Let G be a red graph for w, then if on the web we colour the faces which belongs to $V(G)$, then the external degree of a face f in $V(G)$ is the number of half-edges of w which touch the face f and lie in the uncoloured region. These half-edges are called the *grey half-edges* of f in G or of G when we consider the set of all grey half-edges of all vertices of G. See figure 4.3.

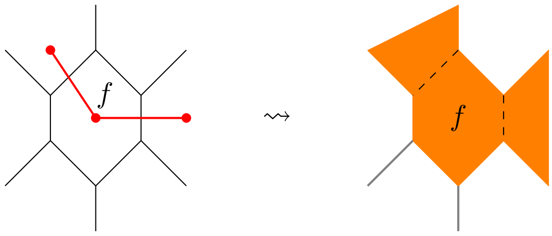


Figure 4.3: Interpetation of the external degree in terms of grey half-edges. On the left, a portion of a web w with a red graph; on the right, the same portion of w with the vertices of G orange-coloured. The external degree of f is the number of half edges touching f which are not orange. In our case $\mathrm{ed}(f) = 2$.

An oriented red graph is a red graph together with an orientation, *a priori* there is no restriction on the orientations, but as we shall see just a few of them will be relevant to consider.

Definition 4.1.4. Let w be an ε-web, G be a red graph for w and o an orientation for G, we define the level $i_o(f)$ (or $i(f)$ when this is not ambiguous) of a vertex f of G by the formula:

$$\begin{aligned} i_o(f) &\stackrel{\text{def}}{=} 2 - \frac{1}{2}\mathrm{ed}(f) - \#\{\text{edges of } G \text{ pointing to } f\} \\ &= 2 - \frac{\deg_{D(w)}}{2} + \#\{\text{edges of } G \text{ pointing away from } f\} \end{aligned}$$

and the level $I(G)$ of G is the sum of levels of all vertices of G.

Remark 4.1.5. The level is an integer because of remark 4.1.3. Note that the level of G does not depend on the orientation of G and we have the formula:

$$I(G) = 2\#V(G) - \#E(G) - \frac{1}{2}\sum_{f \in v(G)} \mathrm{ed}(f).$$

Definition 4.1.6. A red graph is *admissible* if one can choose an orientation such that for each vertex f of G we have: $i(f) \geqslant 0$. Such an orientation is called *a fitting orientation.* An admissible red graph G for w is *exact* if $I(G) = 0$.

Definition 4.1.7. Let w be an ε-web and G be a red graph for w. A *pairing* of G is a partition of the grey half-edges of G into subsets of 2 elements such that for any subset the two half-edges touch the same face f, and one points to f and the other one points away from f. A red graph together with a pairing is called a *paired red graph.*

Definition 4.1.8. A red graph G in an ε-web w is *fair* (resp. *nice*) if for every vertex f of G we have $\mathrm{ed}(f) \leqslant 4$ (resp. $\mathrm{ed}(f) \leqslant 2$).

Lemma 4.1.9. *If G is an admissible red graph in an ε-web w, then G is fair.*

Proof. It follows directly from the definition of the level. □

Corollary 4.1.10. *Let w be a non-elliptic ε web, then if G is an admissible red graph for w then it has at least two edges.*

Proof. If G would contain just one vertex f, this would have external degree greater or equal to 6, contradicting lemma 4.1.9. We can actually show that such a red graph contains at least 6 vertices (see corollary 4.1.20 and proposition 4.1.23). □

Remark 4.1.11. If a red graph G is nice, there is only one possible pairing. If it is fair the number of pairing is 2^n where n denote the number of vertices with external degree equal to 4.

If on a picture one draws together a web w and a red graph G for w, one can encode a pairing of G on the picture by joining[1] with dashed line the paired half-edges. Note that if G is fair it's always possible to draw disjoint dashed lines (see figure 4.4 for an example).

Figure 4.4: A web w, a red graph G and the two possible pairings for G.

The rest of the chapter (respectively in section 4.2 and 4.3) will be devoted to show the following two theorems:

Theorem 4.1.12. *To every exact paired red graph of w we can associate a non trivial idempotent of $Hom_{K^\varepsilon}(P_w, P_w)$. Further more the direct factor associated with the idempotent is a web-module.*

Theorem 4.1.13. *Let w be a non-elliptic ε-web, then if $\langle \overline{w} w\rangle$ is non-monic or have degree bigger than $l(\varepsilon)$, then there exists an exact red graph for w, therefore the K^ε-module P_w is decomposable.*

4.1.2 Combinatorics on red graphs

On the one hand, the admissibility of a red graph relies on the local non-negativity of the level for some orientation, on the other hand the global level I does not depend on the orientation. However, it turns out that the existence of admissible red graph G for an ε-web w can be understood thanks to I in some sense:

Proposition 4.1.14. *Let w be an ε-web, suppose that there exists G a red graph for w such that $I(G) \geqslant 0$, then there exists an admissible red graph $\widetilde{G}$ for w such that $I(\widetilde{G}) \geqslant I(G)$.*

[1] We impose that w intersect the dashed lines only at their ends.

Proof. If G is already admissible, there is nothing to show, hence we suppose that G is not admissible. Among all the orientations for G, we choose one such that $\sum_{f\in V(G)}|i(f)|$ is minimal, we denote is by o. From now on G is endowed with this orientation. As G is not admissible there exists some vertices with negative level and some with positive level.

We first show that there is no oriented path from a vertex f_p with $i_o(f_p) > 0$ to a vertex f_n with $i_o(f_n) < 0$. Suppose there exists γ such a path. Let us inspect o' the orientation which is the same as o expect along the path γ where it is reversed. For all vertices f of G but f_p and f_n, we have $i_o(f) = i_{o'}(f)$ (for all the vertices inside the path, the position of the edges pointing to them is changed, but not their number), and we have:

$$i_{o'}(f_p) = i_o(f_p) - 1 \quad i_{o'}(f_n) = i_o(f_n) + 1.$$

But then $\sum_{f\in V(G)}|i_{o'}(f)|$ would be strictly smaller than $\sum_{f\in V(G)}|i_o(f)|$ and this contradicts that o is minimal.

We consider $(\widetilde{G}, \tilde{o})$ the induced oriented sub-graph of (G, o) with set of vertices $V(\widetilde{G})$ equal to the vertices of G which can be reach from a vertex with positive level by an oriented path. This set is not empty since it contains the vertices with positive degree. It contains no vertex with negative degree. For all vertices of $\widetilde{G}$, we have:

$$\begin{aligned} i_{\tilde{o}}(f) &= 2 - \frac{\deg_{D(w)}(f)}{2} + \#\{\text{edges of } G \text{ pointing away from } f \text{ in } \widetilde{G}\} \\ &= 2 - \frac{\deg_{D(w)}(f)}{2} + \#\{\text{edges of } G \text{ pointing away from } f \text{ in } G\} \\ &= i_o(f). \end{aligned}$$

The second equality holds because if f is in $V(\widetilde{G})$ all the edges in $E(G) \setminus E(G')$. G which are not in $\widetilde{G}$ point to f by definition of $\widetilde{G}$. This shows that $\widetilde{G}$ is admissible and $I(\widetilde{G}) > I(G)$. □

Lemma 4.1.15. *Let w be a non-elliptic web, suppose that it contains a red graph of level k, then it contains an admissible nice red graph of level at least k.*

Proof. We consider G a red graph of w of level k. Thanks to lemma 4.1.14 we can suppose that it is admissible. We can take a minimal red graph G for the property of being of level at least k and admissible. The graph G is endowed with a fitting orientation. Now suppose that it is not nice, it means that there exists a vertex v of G which have exterior degree equal to 4. But G being admissible all the edges of G adjacent to v point out of v, so that we can remove v *i.e.* we can consider then induced sub-graph G' with all the vertex of G but v with the induced orientation. Then it is admissible, with the same level, hence G is not minimal, contradiction. □

For a non-elliptic ε-web, the existence of an exact red graph may appear as an exceptional situation between the case where there is no admissible red graph and the case where all admissible red graphs are non-exact. The aim of the rest of this section is to show the proposition 4.1.16 which indicates that this is not the case. On the way we state some small results which are not directly useful for the proof but may alight what red-graphs look like.

Proposition 4.1.16. *Let w be a non-elliptic ε-web. If there exists an admissible red graph for w then there exists an exact red graph for w.*

Definition 4.1.17. Let w be an ε-web, and G and G' two admissible red graphs for w. We say that G' is a *red sub-graph* of G if $V(G') \subset V(G')$. We denote by $\mathcal{G}(G)$ the set of all admissible red sub-graphs. It is endowed with the order given by the inclusion of sets of vertices. We say that G is *minimal* if $\mathcal{G}(G) = \{G\}$.

Note that a red sub-graph is an induced sub-graph and that a minimal red-graph is connected.

Lemma 4.1.18. *Let w be an ε-web and G a minimal admissible red graph endowed with a fitting orientation. There is no non-trivial partition of $V(G)$ into two sets V_1 and V_2 such that for each vertex v_1 in V_1 and each vertex v_2 in V_2 every edge between v_1 and v_2 is oriented from v_1 to v_2.*

Proof. If there were a such a partition, we could consider the red sub-graph G' with $V(G') = V_2$. For every vertex in V_2 the level is the same in G and in G' and hence, G' would be admissible and G would not be minimal. □

Corollary 4.1.19. *Let w be a ε-web and G a minimal admissible red graph for w, then the graph G has no leaf*[2]*. Therefore if it has 2 or more vertices, then it is not a tree.*

Proof. Indeed, if v were a leaf of G, the vertex v would be either a sink or a source, hence $V(G) \setminus \{v\}$ and $\{v\}$ would partitioned $V(G)$ in a way forbidden by lemma 4.1.18. □

Corollary 4.1.20. *If G is an admissible red graph for a non-elliptic ε-web w, then G is not a tree.*

Proof. Consider a minimal red sub-graph of G. Thanks to corollaries 4.1.10 and 4.1.19, it is not a tree, hence G is not a tree. □

Lemma 4.1.21. *Let w be an ε-web and G a minimal red graph for w. If G has more than 2 vertices, then it is nice.*

Proof. Suppose that we have a vertex v of G with external degree equal to 4. Consider a fitting orientation for G. All edges of G adjacent to v must point out, otherwise the degree of v would be negative. So v would be a sink and, thanks to lemma 4.1.18, this is not possible. □

Lemma 4.1.22. *Let w be a non-elliptic ε-web and G a minimal admissible red graph. If the red graph G is endowed with a fitting orientation, then it is strongly connected.*

The terms *weakly connected* and *strongly connected* are classical in graph theory the first means that the underlying unoriented graph is connected in the usual sense. The second that for any pair of vertices v_1 and v_2, there exists an oriented path from v_1 to v_2 and an oriented path from v_2 to v_1.

Proof. Let v be a vertex of G, consider the subset V_v of $V(G)$ which contains the vertices of G reachable from v by an oriented path. The sets V_v and $V(G) \setminus V_v$ form a partition of $V(G)$ which must be trivial because of lemma 4.1.18, but v is in V_v therefore $V_v = V(G)$, this is true for any vertex v, and this shows that G is strongly connected. □

Proposition 4.1.23. *If G is a red graph for a non-elliptic ε-web w, then any (not-oriented) simple cycle has at least 6 vertices.*

[2] We mean vertex of degree 1.

Proof. Take C a non-trivial simple cycle in G. We consider the collection of faces of w nested by C (this is non empty thanks to condition (iii) of the definition of red graphs). This defines a plane graph H. We define H' to be the graph H with the bivalent vertices smoothed (we mean here that if locally H looks like —•—, then H' looks like ———). An example of this construction is depicted on figure 4.5.

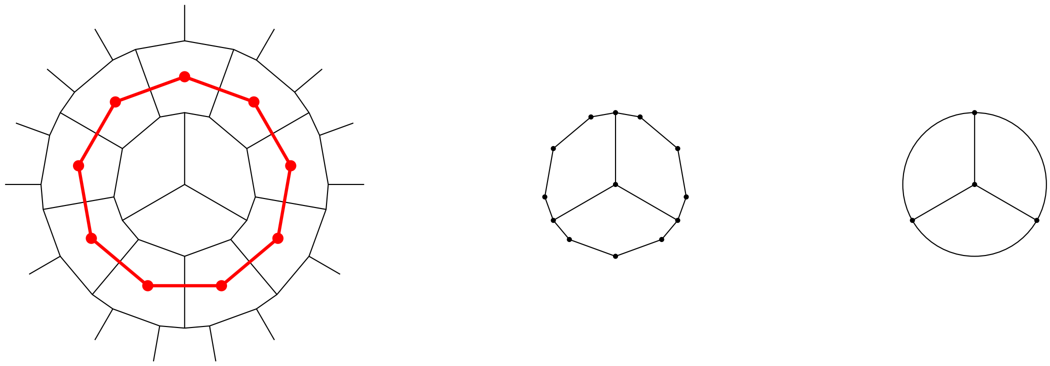


Figure 4.5: On the left the ε-web w and the red graph G, in the middle the graph H, and on the right, the graph H'.

The ε-web w being non-elliptic, each face of H has at least 6 sides. We compute the Euler characteristic of H':

$$\chi(H') = \#F(H') - \#E(H') + \#V(H') = 2.$$

As in proposition 1.1.6, this gives us $\sum_{i\in\mathbb{N}} F_i(H')(1-\frac{i}{6}) = 2$ where $F_i(H)$ is the number of faces of H' with i sides. Restricting the sum to $i \leqslant 5$ and considering F_i' the number of bounded faces, we have:

$$\sum_{i=0}^{5} F_i'(H')(6-i) \geqslant 6.$$

But the bounded faces of H' with less that 6 sides come from bounded faces of H which have at least 6 sides. The number n of bivalent vertices in H is therefore greater than or equal to $\sum_{i=0}^{5} F_i'(H')(6-i)$ *i.e.* greater than or equal to 6. But n is as well the length of the cycle C. □

Note that a cycle in a red graph can have an odd length (as in the example of figure 4.5).

Lemma 4.1.24. *Let G be a minimal admissible red graph for a non-elliptic ε-web w. Then G has at least one vertex with degree 2.*

Proof. Suppose that all vertices of G have degree greater or equal to 3, then the graph G would contain a face with less than 5 sides (this is the same argument than in proposition 1.1.6 which tells that a closed web contains a circle, a digon or a square). But this contradicts lemma 4.1.23. □

The proposition 4.1.16 is a direct consequence of the following lemma:

Lemma 4.1.25. *Let w be a non-elliptic ε-web and G a minimal admissible red graph for w. Then G is exact.*

Proof. We endow G with o a fitting orientation. Suppose G is not exact, then we can find a vertex f with $i_o(f) > 0$.

We first consider the case where $\deg(f) = 2$. The ε-web w being non-elliptic, $\mathrm{ed}(f) \geqslant 2$. This shows that the two edges adjacent to f point away from f, hence, f is a sink and this contradicts lemma 4.1.18.

Now, let us consider the general case. Let f' be a vertex with degree 2. The lemma 4.1.22 implies that there exists γ an oriented path from f to f'. Let us reverse the orientations of the edges of γ. We denote by o' this new orientation. Then we have $i_{o'}(f) = i_o(f) - 1 \geqslant 0$ and $i_{o'}(f') = i_G(f') + 1 \geqslant 1$. The levels of all other edges are not changed, hence o' is a fitting orientation, and we are back in the first situation (where f' plays the role of f). □

4.2 Idempotents from red graphs

Definition 4.2.1. Let w be an ε-web and G a paired red graph for w. We define the G-reduction of w to be the ε-web denoted by w_G and constructed as follows (see figure 4.6 for an example):

1. for every face of w which belongs (as a vertex) to G, remove all edges adjacent to this face.

2. or every face of w connect the grey half-edges of G according to the pairing.

Note that if w is non-elliptic, w_G needs not to be non-elliptic.

Figure 4.6: Example of a G-reduction of an ε-web w. The dotted lines represent the pairing.

Definition 4.2.2. Let w be an ε-web, and G a fair paired red graph for w. We define *the projection associated* with G to be the (w, w_G)-foam denoted by p_G and constructed as follows (from bottom to top):

1. For every edge e' of G, perform an unzip (see figure 4.7) on the edge e corresponding to e in w. Note that the condition (ii) in the definition of red graph implies that all these unzip moves are all far from each other, therefore we can perform all the unzips simultaneously. Let us denote by w' the ε-web at the top of the foam after this step. Each vertex of G corresponds canonically to some a face of w', this faces are circles, digon or square (with an extra information given by the pairing) because G being fair, every vertex of G have an external degree smaller or equal to 4.

2.
   - For each square of w' which corresponds to a vertex of G, perform a square move on it, following the pairing information, (see figure 4.8).
   - For each digon of w' which corresponds to a vertex of G, perform a digon move on it (see figure 4.8).

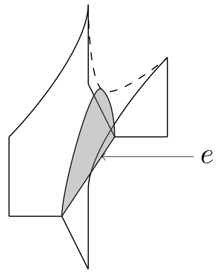


Figure 4.7: Unzip on the edge e.

- For each circle of w' which corresponds to a vertex of G, glue a cap on it (see figure 4.8).

We define as well i_G, *the injection associated with* G to be the (w_G, w)-foam which the mirror image of p_G with respect to the horizontal plane $\mathbb{R}^2 \times \{\frac{1}{2}\}$and $\tilde{e}_G$ to be the (w, w)-foam equal to $i_G \circ p_G$. An example can be seen figure 4.9.

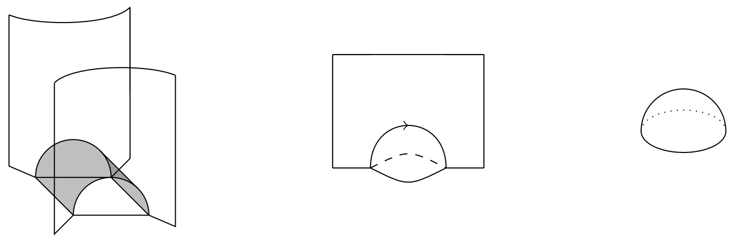


Figure 4.8: A square move, a bigon move and a cap.

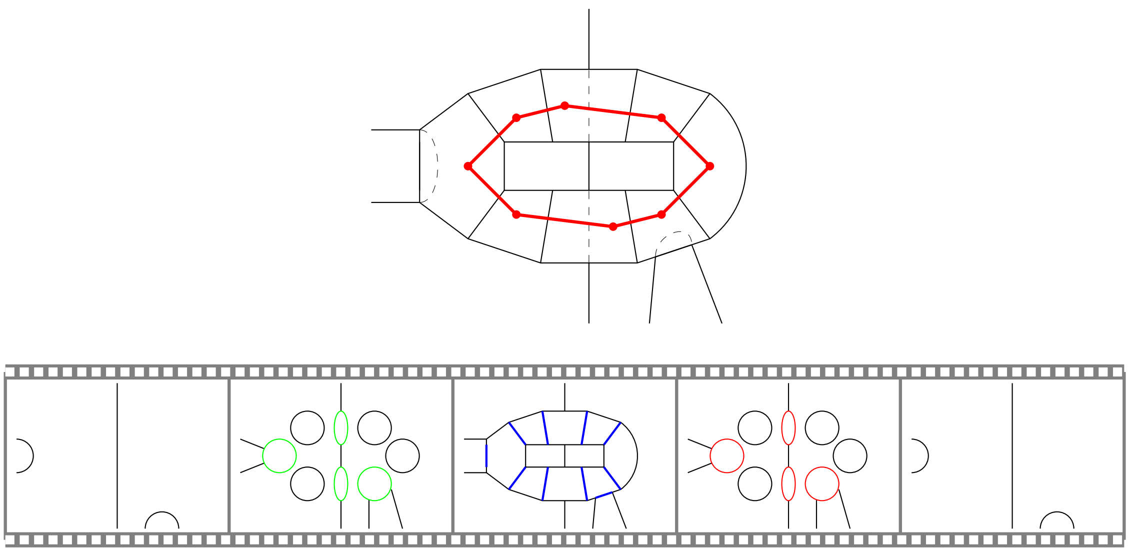


Figure 4.9: On the top a web together with a fair (actually nice) paired red graph G. On the bottom a movie representing e_G.

Remark 4.2.3. It's worthwhile to note that a digon move can be seen as a unzip followed by a cap, and that a square move can be seen as two unzips followed by a cap. With this point of view, we see in that in i_G (and in p_G), every edge of G and every pair of grey half-edges corresponds a zip (or an unzip) and every vertex of G corresponds a cup (or a cup).

The theorem 4.1.12 is an easy consequence of the following proposition:

Proposition 4.2.4. *If w is a non-elliptic web and G is an exact paired red graph for w then the (w_G, w_G)-foam $p_G \circ i_G$ is equivalent under the relations $\boldsymbol{FR}$(see proposition 1.2.11) to a non-zero multiple of the identity (w_G, w_G)-foam $w_G \times [0,1]$.*

To prove this proposition we need to develop a framework to make some calculus with the explicit foams we gave in definition 4.2.2.

4.2.1 Foam diagrams

Definition 4.2.5. Let w be an ε-web, a *foam diagram* κ for w consists of the following data:

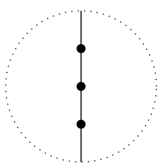
- the ε-web w,
- a fair paired red graph G,
- a function δ (called a *dot function for w*) from $E(w)$ the set of edges of w to $\mathbb{N}$ the set of non-negative integers. This function will be represented by the appropriate number of dots on each edge of w.

With a foam diagram κ we associate $f(\kappa)$ the (w_G, w_G)-foam given by $p_G \circ s_w(\delta) \circ i_G$, where $s_w(\delta)$ is $\mathrm{id}_w = w \times [0,1]$ the identity foam of w with on every facet $e \times [0,1]$ (with $e \in E(w)$) exactly $\delta(e)$ dots. The (w_G, w_G)-foam $f(\kappa)$ is equal to $p_G \circ i_G$, with dots encoded by δ. A foam diagram will be represented by the ε-web drawn together with the red graph, and with some dots added on the edges of the ε-web in order to encode δ.

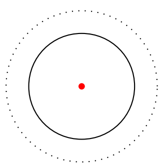
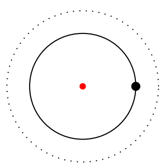
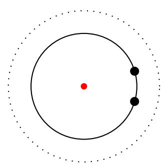
We will often assimilate $\kappa = (w, G, \delta)$ with $f(\kappa)$ and it will be seen as an element of $\hom_{K^\varepsilon}(P_{w_G}, P_{w_G})$. We can rewrite some of the relations depicted on figure 1.11 in terms of foam diagrams:

Proposition 4.2.6. *The following relations on foams associated with foam diagrams hold:*

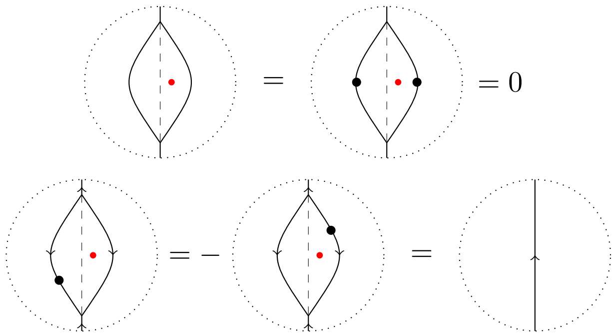
- *The 3-dots relation:*

 $= 0$

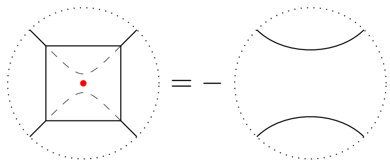
- *The sphere relations:*

 $=$  $= 0$  $= -1$

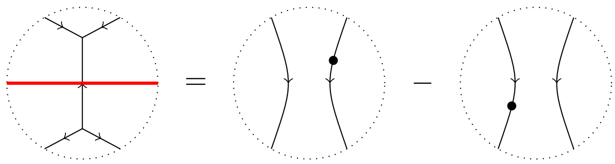
- *The digon relations:*



- *The square relations:*



- *The E-relation:*



The dashed lines indicate the pairing, and when the orientation of the ε-web is not depicted the relation holds for any orientation.

Proof. This is equivalent to some of the relations depicted on figure 1.11. ☐

Lemma 4.2.7. *Let w be an ε-web and $\kappa = (w, G, \delta)$ a foam diagram, with G a fair paired red graph. Then $f(\kappa)$ is equivalent to a $\mathbb{Z}$-linear combination of $s_{w_G}(\delta_i) = f((w_G, \emptyset, \delta_i))$ for δ_i some dots functions for w_G.*

Proof. Thanks to the E-relation of proposition 4.2.6, one can express $f(\kappa)$ as a $\mathbb{Z}$-linear combination of $f((w_j, G_j, \delta_j))$ where the G_j's are red graphs without any edge. Tanks to the sphere, the digon and square relations of proposition 4.2.6, each $f((w_j, G_j, \delta_j))$ is equivalent either to 0 or to $\pm f(w_G, \emptyset, \delta'_j)$. This proves the lemma. ☐

Lemma 4.2.8. *Let w be an ε-web and $\kappa = (w, G, \delta)$ a foam diagram, with G exact, then $f(\kappa)$ is equivalent to a multiple of $w_G \times [0, 1]$.*

Proof. From the previous lemma we know that $f(\kappa)$ is equivalent to a $\mathbb{Z}$-linear combination of $w_G \times [0, 1]$ with some dots on it. We will see that the foam $f(\kappa)$ has the same degree as the foam $w_G \times [0, 1]$. This will prove the lemma because adding a dot on a foam increases its degree by 2.

To compute the degree of $f(\kappa)$ we see it as a composition of elementary foams thanks to its definition:

$$\begin{aligned}\deg f(\kappa) =& \deg(w \times [0,1]) + 2 \cdot \left(2 \cdot \#V(G) - \left(\#E(G) + \frac{\#\{\text{grey half-edges of } G\}}{2}\right)\right) \\ =& |\partial w| + 2 \cdot 0 \\ =& \deg w_G \times [0, 1].\end{aligned}$$

The first equality is due to the decomposition pointed out in remark 4.2.3 and because an unzip (or a zip) has degree -1 and a cap (or a cup) has degree +2. The factor 2 is due to the fact $f(\kappa)$ is the composition of i_G and p_G. The second one follows from the exactness of G. $\square$

To prove the proposition 4.2.4, we just need to show that in the situation of the last lemma, the multiple is not equal to zero. In order to evaluate this multiple, we extend foam diagrams to (partially) oriented paired red graphs by the local relation indicated on figure 4.10.

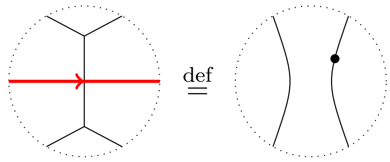


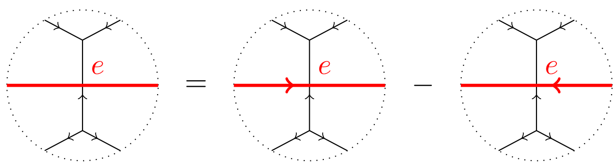
Figure 4.10: Extension of foam diagrams to oriented red graphs.

By "partially oriented" we mean that some edges may be oriented some may not. If G is partially oriented, and κ is a foam diagram with red graph G, we say that κ' is the *classical foam diagram* associated with κ if it obtained from κ by applying the relation of figure 4.10 on every oriented edges. Note that κ and κ' represent the same foam.

Definition 4.2.9. If w is an ε-web, G a red graph for w and o a partial orientation of G we define $\gamma(o)$ to be equal to $\#\{\text{negative edges of } G\}$. A *negative* (or *positive*) edge is an oriented edge of the red graph, and it's negativity (or positivity) is given by figure 4.11.

Figure 4.11: On the left, a positive edge. On the right, a negative edge.

Lemma 4.2.10. *Let w be an ε-web and G a partially oriented red graph with e a non-oriented edge of G, then we have the following equality of foams:*



If G is an un-oriented red graph for w and δ a dots function for w, then:

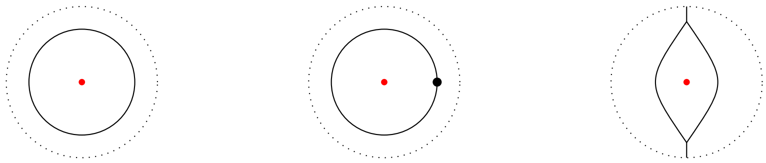
$$f(w, G, \delta) = \sum_o (-1)^{\gamma(o)} f(w, G_o, \delta),$$

where G_o stands for G endowed with the orientation o, and o runs through all the $2^{\#E(G)}$ complete orientations of G.

Proof. The first equality is the translation of the E-relation (see proposition 4.2.6) in terms of foam diagrams of partially oriented red graphs. The second formula is the expansion of the first one to all edges of G. $\square$

Lemma 4.2.11. *If w is an ε-web, G an exact paired red graph for w, o a non-fitting orientation for G and δ the null dot function on w, then the (w_g, w_G)-foam $f(w, G_o, \delta)$ is equivalent to 0.*

Proof. The orientation o is a non-fitting orientation. Hence, there is at least one vertex v of G so that $i_o(v) > 0$. There are two different situations, either $i_o(v) = 1$ or $i_o(v) = 2$. Using the definition of a foam diagrams for oriented red graphs (figure 4.10), we deduce that κ' the classical foam diagram associated with $f(w, G_o, \delta)$ looks around v like one of the three following situations:



The sphere relations and the digon relations provided by proposition 4.2.6 we see that the foam $f(w, G_o, \delta)$ is equivalent 0. □

Lemma 4.2.12. *If w is an ε-web, G an exact paired red graph for w, o a fitting orientation for G and δ the null dots function on w, then the (w_G, w_G)-foam $f(w, G_o, \delta)$ is equivalent to $(-1)^{\mu(o)} w_G \times I$, where $\mu(o) = \#V(G) + \#\{$positive digons of $G_o\}$ (see definition figure 4.12).*

Proof. Let $\kappa' = (w', G', \delta')$ be the classical foam diagram associated with (w, G_o, δ). The red graph G' has no edge. Locally, the foam diagram κ' corresponds to one of the 5 situation depicted on figure 4.12.

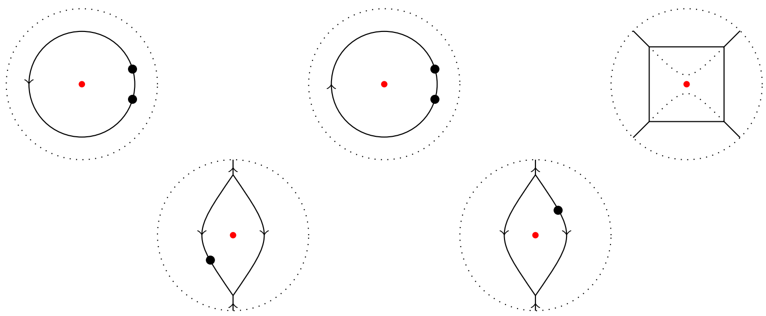


Figure 4.12: The 5 different local situations of a foam diagram κ' next to a vertex of G'. On the second line, the digon on the left is *positive* and the digon on the right is *negative*.

But now using some relations of proposition 4.2.6 we can remove all the vertices of G', we see that $f(w, G_o, \delta)$ is equivalent to $(-1)^{\#V(G') - \#\{\text{positive digons}\}}$ because the positive digon is the only one with no minus sign in the relations of prop 4.2.6. This proves the result because $V(G) = V(G')$. □

Lemma 4.2.13. *If w is an ε-web, G an exact paired red graph for w and o_1 and o_2 two fitting orientations for G, then $\mu(o_1) + \gamma(o_1) = \mu(o_2) + \gamma(o_2)$.*

Proof. We consider $\kappa'_1 = (w', G', \delta_1)$ and $\kappa'_2(w', G', \delta_2)$ the two classical foam diagrams corresponding to (w, G_{o_1}, δ) and (w, G_{o_2}, δ), with δ the null dots function for w.

The red graph G' has no edge, and the local situation are depicted on figure 4.12. Consider v a vertex of G', then a side of the face of w corresponding to v is either clockwise or counterclockwise oriented (with respect to this face). From the definition of γ we obtain that for $i = 1, 2$, $\gamma(o_i)$ is equal to the number of dots in κ'_i on clockwise oriented edges in w'. The dots functions δ_1 and δ_2 differs only next to the digons, so that $\gamma(o_1) - \gamma(o_2)$ is equal to the number of negative digons in κ'_1 minus the number of negative digons in κ'_2. So that we have:

$$\begin{aligned}\gamma(o_1) - \gamma(o_2) &= \mu(o_2) - \mu(o_1)\\ \gamma(o_1) + \mu(o_1) &= \gamma(o_2) + \mu(o_2).\end{aligned}$$

□

Proof of proposition 4.2.4. The foam $p_G \circ i_G$ is equal to $f(w, G, \delta)$ with δ the null dot function on w. From the lemmas 4.2.10, 4.2.11 and 4.2.12 we have that:

$$\begin{aligned}f(w, G, \delta) &= \sum_{o \text{ fitting orientation of } G} (-1)^{\gamma(o)} f(w, G_o, \delta)\\ &= \sum_{o \text{ fitting orientation of } G} (-1)^{\gamma(o)+\mu(o)} w_G \times [0, 1]\\ &= \pm\, \#\{\text{fitting orientations of } G\} w_G \times [0, 1].\end{aligned}$$

The red graph G is supposed to be exact. This means in particular that the set of fitting orientation is not empty. So that $p_G \circ i_G$ is a non-trivial multiple of $\mathrm{id}_{w_G} = w_G \times [0, 1]$. □

Proof of theorem 4.1.12. From the proposition 4.2.4, we know that there exists a non zero integer λ_G such that $p_G \circ i_G = \lambda_G w_G$. Hence, $\frac{1}{\lambda_G} i_G \circ p_G$ is an idempotent. It's clear that it's non-zero. It is quite intuitive that it is not equivalent to the identity foam, for a proper proof, see proposition 4.2.15. □

4.2.2 On the identity foam

Definition 4.2.14. Let w be an ε-web, and f a (w, w)-foam, we say that f is *reduced* if every facet of f is diffeomorphic to a disk and if f contains no singular circle (*i.e.* only singular arcs). In particular this implies that every facet of f meets $w \times \{0\}$ or $w \times \{1\}$.

The aim of this section is to prove the following proposition:

Proposition 4.2.15. *Let w be a non-elliptic ε-web. If f is a reduced (w, w)-foam which is equivalent (under the foam relations **FR**) to a non-zero multiple of $w \times [0, 1]$, then the underlying pre-foam is diffeomorphic to $w \times [0, 1]$ and contains no dot.*

For this purpose we begin with a few technical lemmas:

Lemma 4.2.16. *Let w be a closed web and e an edge of w. Then there exists f a $(\emptyset, w)$-foam which is not equivalent to 0 such that the facet of f touching the edge e contains at least one dot.*

Proof. We prove the lemma by induction on the number of edges of the web w. It is enough to consider the case w connected because the functor $\mathcal{F}$ is monoidal. If the web w is a circle this is clear, since a cap with one dot on it is not equivalent to 0. If w is the theta web, then this is clear as well, since the half theta foam with one dot on the facet meeting e is not equivalent to 0.

Else, there exists a square or digon in w somewhere far from e. Let us denote w' the web similar to w but with the digon replaced by a single strand or the square smoothed in one way or the other. By induction we can find an $(\emptyset, w')$-foam f' non-equivalent to 0 with one dot on the facet touching e.

Next to the strand or the smoothed square, we consider a digon move or a square move (move upside down the pictures of figure 4.8). Seen as a (w', w)-foam it induces an injective map. Therefore, the composition of f' with this (w', w)-foam is not equivalent to 0 and has one dot on the facet touching e. $\square$

Notation 4.2.17. Let w be an ε-web, and e be an edge of w. We denote by $f(w, e)$ the $(\emptyset, \overline{w}w)$-foam which is diffeomorphic to $w \times [0, 1]$ with one dot on the facet $e \times [0, 1]$. We denote by $f(w, \emptyset)$ the $(\emptyset, \overline{w}w)$-foam which is diffeomorphic to $w \times [0, 1]$ with no dot on it.

Corollary 4.2.18. *Let w be an ε-web, and e an edge of w, then $f(w, e)$ is non-equivalent to 0.*

Proof. From lemma 4.2.16 we know that for any w, there exists a (w, w)-foam which is non equivalent to 0 and is the product of $f(w, e)$ with another (w, w)-foam. This proves that the (w, w)-foam $f(w, e)$ is not equivalent to 0. $\square$

Definition 4.2.19. If w is an ε-web. We say that it contains a λ (resp. a $\cap$, resp. a H) if next to the border w looks like one of the pictures of figure 4.13.

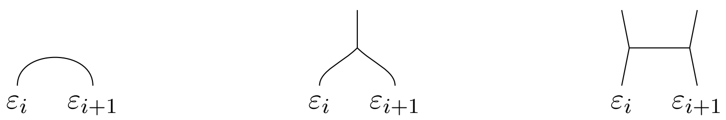


Figure 4.13: From left to right: a λ, a $\cap$ and a H.

Lemma 4.2.20. *Every non-elliptic ε-web contains at least a λ, a $\cap$ or an H.*

Proof. The closed web $\overline{w}w$ contains a circle a digon or a square, and this happens only if w contains a $\cap$ a λ or a H. $\square$

Remark 4.2.21. In fact, one can "build" every non-elliptic web with this three elementary webs. This is done via the "growth algorithm" (see [KK99]).

Lemma 4.2.22. *Let w be a non-elliptic ε-web. Then the elements of $(f(w, e))_{e \in E(w)}$ are pairwise non-equivalent (but they may be linearly dependant).*

Sketch of the proof. We proceed by induction on the number of edges of w. The initiation is straightforward since if w has only one edge there is nothing to prove. We can distinguish several case thanks to lemma 4.2.20:

If w contains a $\cap$, we denote by e the edge of this $\cap$, and by w' the ε'-web similar to w but with the cap removed. Suppose that $e_1 = e$, then $e_2 \neq e$ and, then the $(\emptyset, \overline{w}w)$-foams $f(w, e_1)$ and $f(w, e_2)$ are different because if we cap the cup (we mean $e \times I$) by a cap with one dot on it, on the one hand we obtain a $(\emptyset, \overline{w'}w')$-foam equivalent to 0 and on the other hand a $(\emptyset, \overline{w'}w')$-foam equivalent to $f(w', \emptyset)$. Thanks to lemma 4.2.16, we know that this last $(\emptyset, \overline{w'}w')$-foam is not equivalent to 0. If e_1 and e_2 are different from e, it is clear as well, because $f(w, e_1)$ and $f(w, e_2)$ can be seen as compositions of $f(w', e_1)$ and $f(w', e_2)$ with a birth (seen as a $(\overline{w'}w', \overline{w}w)$-foam) which is known to correspond to injective map.

This is the same kind of argument for the two other cases. The digon relations and the square relations instead of the sphere relations. $\square$

Lemma 4.2.23. *Let w be an ε-web and f a reduced (w, w)-foam f. Suppose that every facet touches $w \times \{0\}$ on at most one edge, and touches $w \times \{1\}$ on at most one edge, then it is isotopic to $w \times [0, 1]$.*

Proof. The proof is inductive on the number of vertices of w. If w is a collection of arcs, the foam f has no singular arc. As f is supposed to be reduced, it has no singular circle. Therefore it is a collection of disks which corresponds to the arcs of w, and this proves the result in this case.

We suppose now that w has at least one vertex. Let us pick a vertex v which is a neighbour (via an edge that we call e) of the boundary ε of w. We claim that the singular arc α starting at $v \times \{0\}$ must end at $v \times \{1\}$.

Indeed, the arc α cannot end on $w \times \{0\}$, for otherwise, the facet f touching e would touch another edge of w. Therefore the arc α ends on $w \times \{1\}$. For exactly the same reasons, it has to end on $v \times \{1\}$, so that the facet which touches $e \times \{0\}$ is isotopic to $e \times I$, now we can remove a neighbourhood of this facet and we are back in the same situation with a ε'-web with less vertices, and this concludes. □

Proof of proposition 4.2.15. We consider w a non-elliptic ε-web. Let f be a reduced (w, w)-foam such that f is equivalent to $w \times I$ up to a non-trivial scalar. Because of lemma 4.2.22, the foam f satisfies the hypotheses of lemma 4.2.23, so that f is isotopic to $w \times [0, 1]$. □

We conjecture that the proposition 4.2.15 still holds without the non-ellipticity hypothesis. However the proof has to be changed since lemma 4.2.22 cannot be extend to elliptic webs (consider the facets around a digon).

Corollary 4.2.24. *If w is a non-elliptic ε-web and w' is an ε-web with strictly less vertices than w, then if f is a (w, w')-foam and g is a (w', w)-foam, then the (w, w)-foam fg cannot be equal to a scalar times the identity.*

4.3 Characterisation of indecomposable web-modules

4.3.1 General View

The lemma 2.2.22 states that the indecomposability of a web-modules P_w can be deduced from the Laurent polynomial $\langle \overline{w} w \rangle$. In this section we will show a reciprocal statement. We first need a definition:

Definition 4.3.1. Let ε be an admissible sequence of signs of length n, an ε-web w is said to be *virtually indecomposable* if $\langle \overline{w} w \rangle$ is a monic symmetric Laurent polynomial of degree n. An ε-web which is not virtually indecomposable is *virtually decomposable.* If w is a virtually decomposable ε-web, we define the *level of* w to be the integer $\frac{1}{2}(\deg \langle \overline{w} w \rangle - n)$.

Despite of its fractional definition, the level is an integer. With this definition, lemma 2.2.22 can be rewritten:

Lemma 4.3.2. *If w is a virtually indecomposable ε-web, then $M(w)$ is an indecomposable K^ε-module.*

The purpose in this section is to prove a reciprocal statement in order to have:

Theorem 4.3.3. *Let ε be an admissible sequence of signs of length n, and w an ε-web. Then the K^ε-module P_w is indecomposable if and only if w is virtually indecomposable.*

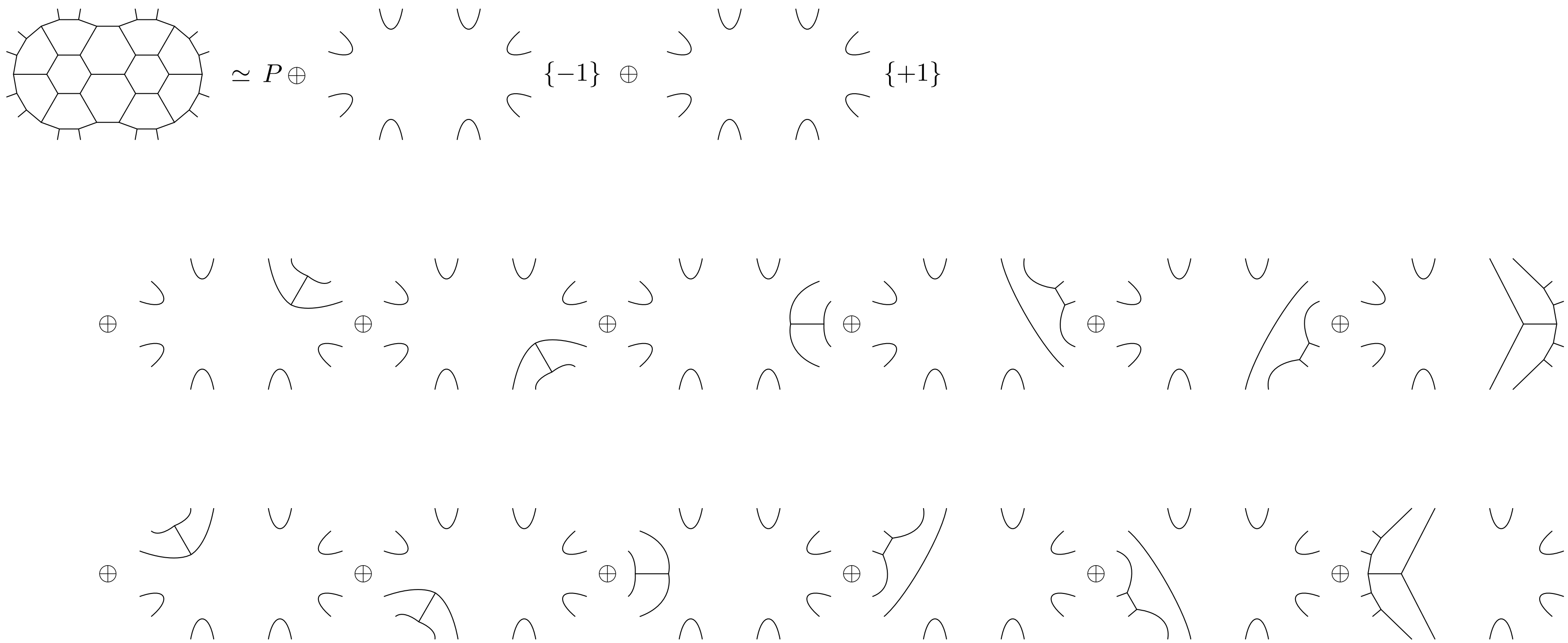


Figure 4.14: Example of a decomposition of a web-module into indecomposable modules. All direct factors which are web-modules are obtained through idempotents associated with red graphs. The module P is not a web-module but is a projective indecomposable module.

Remark 4.3.4. Note that we do not suppose that w is non-elliptic, but as a matter of fact, if w is elliptic then $\langle \overline{w}w \rangle$ is not monic of degree n and the module P_w is decomposable.

To prove the unknown direction of theorem 4.3.3 we use red graphs developed in the previous section and will show a more precise version of the theorem:

Theorem 4.3.5. *If w is a non-elliptic virtually decomposable ε-web of level k, then w contains an admissible red graph of level k, hence* $\mathrm{End}_{K^\varepsilon}(P_w)$ *contains a non-trivial idempotent and P_w is decomposable.*

Proof of theorem 4.3.3 assuming theorem 4.3.5. Let w be a virtually decomposable ε-web and let us denote by k its level. From theorem 4.3.5 we know that there exists a red graph G'' of level k. But then, thanks to proposition 4.1.14, there exists G' a sub red graph of G'' which is admissible. And finally, the proposition 4.1.16 shows the existence of an exact red graph G in w. We can apply theorem 4.1.12 to G and this tells that P_w is decomposable. □

The proof of theorem 4.3.5 is a recursion on the number of edges of the web w. But for the recursion to work, we need to handle elliptic webs as well. We will actually show the following:

Proposition 4.3.6.

1. *If w is a ∂-connected ε-web which is virtually decomposable of level $k \geqslant 1$ then there exists S a stack of nice red graphs for w of level greater or equal to k such that w_S is ∂-connected.*

2. *If w is a ∂-connected ε-web which is virtually decomposable of level $k \geqslant 1$, contains no digon and contains exactly one square which is supposed to be adjacent the unbounded face then there exists a nice red graph G in w of level greater or equal to k such that w_G is ∂-connected.*

3. *If w is a non-elliptic ε-web which is virtually decomposable of level $k \geqslant 0$ then there exists a nice red graph G in w of level greater of equal to k such that w_G is ∂-connected.*

Before proving the proposition we need to introduce *stacks of red graphs* (see below), and the notion of ∂-connectedness (see section 4.3.2). Then we will prove the proposition 4.3.6 thanks to a technical lemma (lemma 4.3.16) which will be proven in section 4.3.5 after an alternative glance on red graphs (section 4.3.4).

Remark 4.3.7. It is easy to see that a non-elliptic superficial ε-web contains no red graphs of non-negative level, hence this result is strictly stronger than the theorem 3.2.4.

Definition 4.3.8. Let w be an ε-web, *a stack of red graphs $S = (G_1, G_2, \ldots, G_l)$ for w* is a finite sequence of paired red graphs such that G_1 is a red graph of $w_1 \stackrel{\text{def}}{=} w$, G_2 is a red graph of $w_2 \stackrel{\text{def}}{=} w_{G_1}$, G_3 is a red graph of $w_3 \stackrel{\text{def}}{=} (w_{G_1})_{G_2} = (w_2)_{G_2}$ etc. We denote $(\cdots((w_{G_1})_{G_2})\cdots)_{G_l}$ by w_S and we denote l by $l(S)$ and we say that it is *the length of S*. We define the level of a stack to be the sum of the levels of the red graphs of the stack.

Definition 4.3.9. A stack of red graphs is *nice* if all its red graphs are nice. Note that in this case the pairing condition on red graphs is empty.

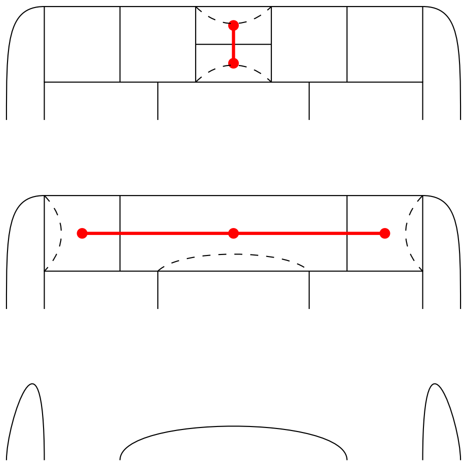


Figure 4.15: A stack of red graphs of length 2.

4.3.2 The ∂-connectedness

Definition 4.3.10. An ε-web is *∂-connected* if every connected component of w touches the border.

A direct consequence is that a ∂-connected ε-web contains no circle.

Lemma 4.3.11. *A non-elliptic ε-web is ∂-connected.*

Proof. An ε-web which is not ∂-connected has a closed connected component, this connected component contains at least a circle, a digon or a square and hence is elliptic. □

Lemma 4.3.12. *Let w be a ∂-connected ε-web with a digon, the web ε-web w' equal to w except that the digon reduced (see figure 4.16) is still ∂-connected. In other words ∂-connectedness is preserved by digon-reduction.*

Figure 4.16: On the left w, on the right w'.

Proof. This is clear because every path in w can be projected onto a path in w'. □

Note that ∂-connectedness is not preserved by square reduction, see for example figure 4.17. However we have the following lemma:

Figure 4.17: The ∂-connectedness is not preserved by square reduction.

Lemma 4.3.13. *If w is a ∂-connected ε-web which contains a square S then one of the two ε-webs obtained from w by a reduction of S (see figure 4.18) is ∂-connected.*

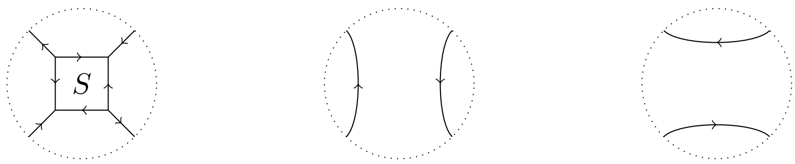


Figure 4.18: On the right the ε-web w with the square S, on the middle and on the right, the two reductions of the square S.

Proof. Consider the oriented graph $\tilde{w}$ obtained from w by removing the square S and the 4 half-edges adjacent to it (see figure 4.19). We obtain a graph with 4 less cubic vertices

Figure 4.19: On the left w, on the right $\tilde{w}$.

than w and 4 more vertices of degree 1 than w. We call E_S the cyclically ordered set of the 4 vertices of $\tilde{w}$ of degree 1 next to the removed square S. The orientations of the vertices in E_S are $(+,-,+,-)$. Note that in $\tilde{w}$, the flow modulo 3 is preserved everywhere, so that the sum of orientation of vertices of degree 1 of any connected component must be equal to 0 modulo 3. Suppose now that there is a connected component t of $\tilde{w}$ which has all its vertices of degree 1 in E_S, the flow condition implies that either all vertices of E_S are vertices of t or exactly two consecutive vertices of E_S are vertices of t, or that t has no vertex of degree 1. The first situation cannot happen because by adding the square to t we would construct a free connected component of w which is supposed to be ∂-connected, the last situation neither for the same reason. So the only thing that can happen is the second situation. If there were two different connected components t_1 and t_2 of $\tilde{w}$ such that t_1 and t_2 have all their vertices of degree 1 in E_S, then adding the square to $t_1 \cup t_2$ would lead to a free connected component of w, so their is at most one connected component of $\tilde{w}$ with all this vertex of degree 1 in E_S call this vertices e_+ and e_-, and call e'_+ and e'_- the two other vertices of E_S (the indices gives the orientation). If we choose w' to be the ε-web corresponding to the smoothing which connects e_+ with e'_- and e_- with e'_+, then w' is ∂-connected. □

Definition 4.3.14. Let w be a ∂-connected ε-web and S a square in w. The square S is *a ∂-square* if the two ε-webs $w_=$ and $w_{||}$ obtained from w by the two reductions by the square S are ∂-connected.

Lemma 4.3.15. *If w is a ∂-connected web, then either it is non-elliptic, or it contains either a digon or a ∂-square.*

Proof. Suppose that w is not non-elliptic. As w is ∂-connected it contains no circle. If must contains at least a digon or a square, if it contains a digon we are done, so suppose w contains no digon. We should show that at least one square is a ∂-square. Suppose that there is no ∂-square, it means that for every square S, there is a reduction such that the ε-web resulting $w_{s(S)}$ obtained by replacing w by the reduction has a free connected component t_S. Let us consider a square S_0 such that t_{S_0} is as small as possible (in terms of number of vertices for example). The web t_{S_0} is closed and connected, so that either it is a circle, or it contains a digon or at least two square. If t_{S_0} is a circle then w contains a digon just next to the square S_0, and we excluded this case (see figure 4.20). If it contains

Figure 4.20: on the left w_{S_0}, on the right w. If t_{S_0} is a circle, then w contains a digon.

a digon, the digon must be next to where S_0 was smoothed else the digon would already be in w. It appears hence that the digon comes from a square S_1 in w (S_1 is adjacent to S_0), and t_{S_1} has two vertices less than T_{S_0} which is excluded (see figure 4.21). The closed web

Figure 4.21: On the left w_{S_0}, on the right w. If t_{S_0} contains a digon then w contains a square adjacent to S_0.

t_{S_0} contains at least two squares so that we can pick up one, we denote it by S', which is far from S_0 and hence comes from a square in w. Now at least one of the two smoothings of the square S' must disconnect t_{s_0} else the square S' would be a ∂-square in w. But as it disconnects t_{S_0}, $t_{S'}$ is a strict sub graph of t_{S_0}, and this contradict the minimality of S_0. And this concludes that w must contain a ∂-square. □

4.3.3 Proof of proposition 4.3.6

In this section we prove the proposition 4.3.6 admitting the following technical lemma:

Lemma 4.3.16. *Let w be a ∂-connected ε-web which contains, no digon and one square which touches the unbounded face. Let G be a nice red graphs of w and G' a nice red graph of w_G such that w_G and $w_{G'}$ are ∂-connected, then there exists G'' a red graph of w such that $(w_G)_{G'} = w_{G''}$ and the level of G'' is bigger or equal to the level of G plus the level of G'.*

This lemma says that under certain condition one can "flatten" two red graphs.

Proof of proposition 4.3.6. As we announced this will be done by recursion on the number of edges of w. We supposed than 1, 2 and 3 hold for all ε-webs with strictly less than n edges, and we consider an ε-web with n edges. Note that whenever w is non-elliptic the statement 3 is stronger than the statement 1, so that we won't prove 1 in this case. We first prove 1:

If w contains a digon, then we apply the result on w' the ε-web similar to w but with the digon reduced (*i.e.* replaced by a single strand). The red graph G which consist of only one edge (the digon) and no edge is nice and has level equal to 1 (see figure 4.22).

Figure 4.22: On the left w', on the right w with the red graph G.

If w' is not virtually decomposable or virtually decomposable of level 0, then w is virtually decomposable of level 1. In this case, the stack with only one red graph equal to G is convenient and we are done. Else we know that w' is of level $k-1$ and that there exists a nice stack of red graphs S' of level $k-1$ in w' and we consider the stack S equal to the concatenation of G with S', it is a nice stack of red graphs of level k and we are done.

Suppose now that the ε-web w contains no digon, but a square, then it contains a ∂-square (see lemma 4.3.15). Suppose that the level of w is $k \geqslant 1$ (else there is nothing to show), then at least one of the two reductions is virtually decomposable of level k (see 1.1.19). Then we consider w' the ε-web obtained by a reduction of the square so that it is of level k. From the induction hypothesis we know that there exists a stack of red graphs S' in w' of level k. If all the red graphs of S' are far from the location of the square, then we can transform the stack S' into a stack of w with the same level. Else, we consider G' the first red graph of S' which is close from the square location and according to the situation we define G by the moves given on figure 4.23.

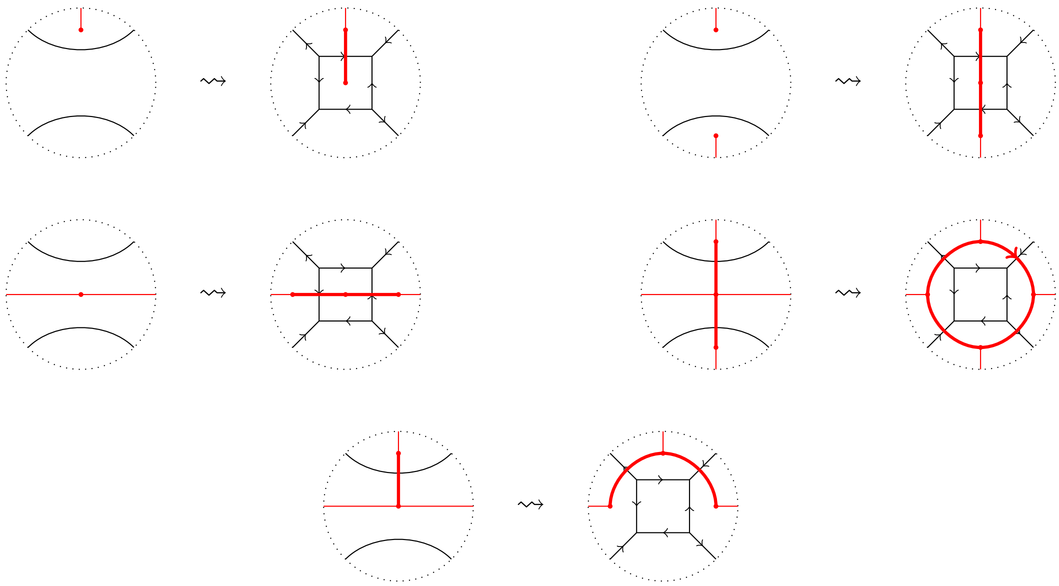


Figure 4.23: Transformations of G' to obtain G.

Replacing G' by G we can transform, the stack S' into a stack for the ε-web w.

We now prove 2.

From what we just did, we know that w contains a nice stack of red graphs of level k. Among all the nice stacks of red graphs of w with level greater or equal to k, we choose one with a minimal length, we call it S. If its length were greater or equal to 2, then lemma 4.3.16 would tell us that we could take the first two red graphs and replace them by just one red graph with a level bigger or equal to the sum of their two levels, so that S would not be minimal, this prove that S has length 1, therefore, w contains a nice red graph of level at least k.

We now prove 3.

The border of w contains at least a $\cap$, a λ, or an H (see figure 4.13). In the two first cases, we can consider w' the ε-web with the $\cap$ removed or the λ replaced by a single strand, then w' is non-elliptic and virtually decomposable of level k and there exists a nice red graph in w' of level at least k, this red graph can be seen as a red graph of w, and we are done. If the border of w contains no λ and no $\cap$, then it must contains an H. There are two ways to reduce the H (see figure 4.24). At least one of the two following situation happens: $w_{||}$ is virtually decomposable of level k or $w_{\_}$ is virtually decomposable of level $k+1$. In the first situation, one can do the same reasoning as before: $w_{||}$ being non-elliptic,

Figure 4.24: The H of w (on the left) and its two reductions: $w_{||}$ (on the middle) and $w_{=}$ (on the right).

the induction hypothesis gives that we can find a nice red graph of level at least k in $w_{||}$, this red graph can be seen as a red graph of w and we are done. In the second situation, we consider $w_{\_}$, we can apply the induction hypothesis to $w_{\_}$ (we are either in case 2 or in case 3), so we can find a nice red graph of level at least $k+1$, coming back to H this gives us a red graph of level at least k (but maybe not nice), and we can conclude via the lemma 4.1.15. □

4.3.4 A new approach to red graphs.

In this section we give an alternative approach to red graphs: instead of starting with a web and simplifying it with a red graph we construct a red graph from a web and a simplification of this web. For this we need a property of webs that we did not use so far.

Proposition 4.3.17. *Let w be a closed web, then it admits a (canonical) face-3-colouring with the unbounded face coloured $c \in \mathbb{Z}/3\mathbb{Z}$. We call this colouring* the face-colouring of base *c of w. When c is not mentioned it is meant to be* 0.

Proof. We will colour connected components of $\mathbb{R}^2 \setminus w$ with elements of $\mathbb{Z}/3\mathbb{Z}$. We can consider the only unbounded component U of $\mathbb{R}^2 \setminus w$. We colour it by c, then for each other connected component f, we consider p an oriented path from a point inside U to a point inside f, which crosses the w transversely, we then define the colour of f to be the sum (modulo 3) of the signs of the intersection of the path p with w (see figure 4.25 for signs convention). This does not depend on the path because in w the flow is always preserved modulo 3. And, by definition, two adjacent faces are separated by an edge, so that they do not have the same colour. □

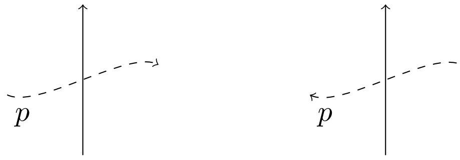


Figure 4.25: On the left a positive crossing, on the right a negative one. The path is dashed and the web is solid.

Corollary 4.3.18. *Let w be an ε-web, then the connected component of $\mathbb{R} \times \mathbb{R}_+ \setminus w$ admits a (canonical) 3-colouring with the unbounded connected component coloured by c. We call this colouring* the face-colouring of base c *of w.*

Proof. We complete w with $\overline{w}$ and we use the previous proposition to obtain a colouring of the faces. This gives us a canonical colouring for $\mathbb{R} \times \mathbb{R}_+ \setminus w$. □

Note that in this corollary it is important to consider the connected component of $\mathbb{R} \times \mathbb{R}_+ \setminus w$ instead of the connected component of $\mathbb{R}^2 \setminus w$. Let us formalise this in a definition.

Definition 4.3.19. If w is an ε-web, the *regions* of w are the connected components of $\mathbb{R} \times \mathbb{R}_+ \setminus w$. The *faces* of w are the regions which do not intersect $\mathbb{R} \times \{0\}$.

Definition 4.3.20. Let w be an ε-web, an ε-web w' is a *simplification of w* if

- the set of vertices of w' is included in the set of vertices of w,
- every edge e of w' is divide into an odd number of intervals $([a_i, a_{i+1}])_{i \in [0,2k]}$ such that for every i in $[0, k]$, $[a_{2i}, a_{2i+1}]$ is an edge of w (with matching orientations) and for every i in $[0, k-1]$, $[a_{2i+1}, a_{2i+2}]$ lies in the faces of w opposite to $[a_{2i}, a_{2i+1}]$ with respect to a_{2i+1} (see figure 4.26).

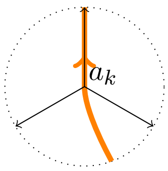


Figure 4.26: Local picture around a_k. The edge of w' is orange and large, while the ε-web w is black and thin.

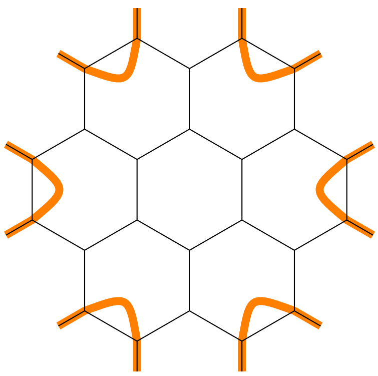


Figure 4.27: The ε-web w (on black) and w_0 (in orange) of proposition 3.1.1 seen in terms of simplification.

Lemma 4.3.21. *Let w be a ε-web and w' a ∂-connected simplification of w If e is an edge of w which is as well a (part of an) edge of w', then in the face-colourings of base c of w and w', the the regions adjacent to e in w and in w' are coloured in the same way.*

Proof. This is an easy recursion on how e is far from the border. □

Note that in this definition the embedding of w' with respect to w is very important.

Definition 4.3.22. Let w be an ε-web and w' a simplification of w. We consider the face-colourings of w and w'. A face f of w lies in one or several regions of w'. This face f is *essential with respect to* w' if all regions of w' it intersects do not have the same colour as f.

Remark 4.3.23. We could have write this definition with region of w instead of faces, but it is easy to see that a region of which is not a face w is never essential.

Lemma 4.3.24. *Let w be a ∂-connected ε-web and w' a ∂-connected simplification of w. If a face f of w is not essential with respect to w' then it intersects only one region of w'.*

Proof. Consider a face f of w which intersects more than one region of w'. We will prove that it is essential with respect to w'. Consider an edge e' of w' which intersects f (there is at least one by hypothesis), when we look next to the border of f next to e' we find a vertex v of w (see figure 4.28).

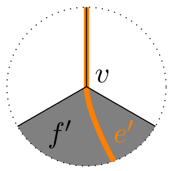


Figure 4.28: A part of the face f' next to an edge e' of w'. Above v the colours of w and w' are coherent thanks to lemma 4.3.21.

We want to prove that none of the faces of w' which are adjacent to e' has the same colour as the face f. This follows from the lemma 4.3.21, and from the fact that the part of e' above v is an edge of w (see figure 4.28). □

Corollary 4.3.25. *Let w be a ∂-connected ε-web and w' a ∂-connected simplification of w. If a face f of w intersects a region of w' which has the same colour, it is not essential.*

Proposition 4.3.26. *Let w be a ∂-connected ε-web (this implies that every face of w is diffeomorphic to a disk) and w' a ∂-connected simplification of w. Then there exists a (canonical) paired red graph G such that w' is equal to w_G. We denote it by $G_{w\to w'}$.*

Proof. We consider the canonical colourings of the faces of w and w'. The red graph G is the induced sub-graph of $w^\star$ (the dual graph of w) whose vertices are essential faces of w with respect to w'. The pairing is given by the edges of w'. We need to prove first that this is indeed a red graph, and in a second step that $w_G = w'$. We consider a vertex v of w and the 3 regions next to it. We want to prove that at least one of the 3 regions is not essential with respect to w'. If the vertex v is a vertex of w' then lemma 4.3.21 and corollary 4.3.25 give that none of the three regions is essential. Else, v either lies inside an edge of w' or it lies in a face of w' (see figure 4.29).

Consider the first situation: one of the 3 regions intersects two different regions of w' hence it is essential thanks to lemma 4.3.24, the two others are not thanks to corollary 4.3.25.

In the last situation, the 3 regions have different colours so that one of them has the same colour than the colour of the region of w' where v lies in, this region is therefore not

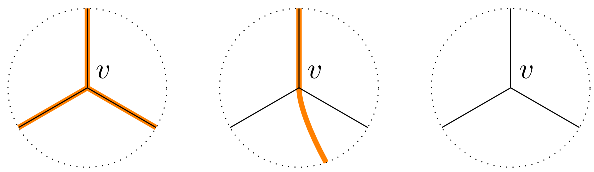


Figure 4.29: The three configurations for the vertex v of w: it is a vertex of w' (on the left), it lies inside an edge of w' (on the middle), it lies inside a region of w' (on the right).

essential (corollary 4.3.25). This shows that G is a red graph (we said nothing about the admissibility).

Let us now show that $w' = w_G$. We consider a collection $(N_f)_{f\in V(G)}$ of regular neighbourhoods of essential faces of w with respect to w'. Let us first show that for every essential face f of w, if N_f is a regular neighbourhood of f, the restriction of w_G and w' matchs. As f is essential it is a vertex of G. Then the restriction of w_G to N_f is just a collection of strands joining the border to the border, just as w'.

In $\mathbb{R}\times\mathbb{R}_+ \setminus \left(\bigcup_{f\in V(G)} N_f\right)$ the ε-webs w' and w_G are both equal to w. This complete the proof. □

Note that $G_{w\to w'}$ depends on how w' is embedded to see it as a simplification of w.

Definition 4.3.27. Let w a ε-web and w' a simplification of w, then the simplification is *nice*, if for every region r of w, $r\cap w'$ is either the empty set or connected.

We have the natural lemma:

Lemma 4.3.28. *Let w be a ∂-connected ε-web and w' a ∂-connected simplification of w. The simplification is nice if and only if the red graph $G_{w\to w'}$ is nice*

Proof. Thanks to lemma 4.3.24, only essential faces of w with respect to w' can have non trivial intersection with w', and for an essential face f, twice the number of connected component of $f\cap w'$ is equal to the exterior degree of the vertex of $G_{w\to w'}$ corresponding to f. □

Lemma 4.3.29. *If w is a ∂-connected ε-web, and w' is a ∂-connected simplification of w. Then the level of $G_{w\to w'}$ is given by the following formula:*

$$i(G_{w\to w'}) = 2\#\{\textit{essential faces of } w \textit{ wrt. } w'\} - \frac{\#V(w) - \#V(w')}{2}.$$

This shows that the embedding of w' influences the level of $G_{w\to w'}$ only on the number of essential faces of w with respect to w'

Proof. The level of a red graph G is given by:

$$i(G) = 2\#V(G) - \#E(G) - \sum_{f\in V(G)} \frac{\mathrm{ed}(f)}{2}.$$

By definition of $G_{w\to w'}$, we have:

$$\{\text{essential faces of } w \text{ wrt. } w'\} = V(G_{w\to w'}).$$

The only thing to realise is that we have:

$$2\left(\#E(G)_{w\to w'} + \sum_{f\in V(G_{w\to w'})} \frac{\mathrm{ed}(f)}{2}\right) = \#V(w) - \#V(w'),$$

and this follows from the definition of $w_{G_{w\to w'}} = w'$. $\square$

Definition 4.3.30. If f is a face of w, w' a simplification of w and r a region of w', we say that f avoids r if $f \cap r = \emptyset$ or if the boundary of r in each connected component of $f \cap r$ joins two consecutive vertices of f (see figure 4.30). In the first case we say that f avoid r trivially. If f is an essential face of w with respect to w' and r is a region of w', we

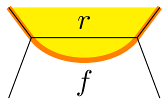


Figure 4.30: The local picture of a face f (in white) of w (in black) non-trivially avoiding a region r (in yellow) of w' (in orange).

say that f *fills* r, if f does not avoid r. If F' is a set of region of w' we say that f *fills (resp. avoids)* F' if it fills at least one region of F' (resp. avoids all the regions of F'). We define:

$$n(f, F') \stackrel{\text{def}}{=} \#\{r \in F' \text{such that } f \text{ fills } r\}.$$

If G' is a red graph of w', we write $n(f, G')$ for $n(f, V(G'))$.

With the same notations, and with F a set of face of w, we have the following equality:

$$\#F = \#\{\text{faces } f \text{ of } F \text{ avoiding } F'\} + \sum_{f'\in F'} \sum_{\substack{f\in F \\ f \text{ fills } f'}} \frac{1}{n(f, F')}. \tag{4.1}$$

Lemma 4.3.31. *Let w be a ∂-connected ε-web and w' a nice ∂-connected simplification of w. Let F' be a collection of faces of w', then for every face f of w, we have: $n(f, F') \leqslant 2$.*

Proof. This is clear since $f \cap w'$ consists of at most one strand, so that it intersects at most 2 faces of F'. $\square$

Remark 4.3.32. Let w be a ε-web, w' a nice ∂-connected simplification of w and f an essential face of w with respect to w'. Suppose that f has at least 6 sides of w. Suppose furthermore that it intersects two regions r_1 and r_2 of w', then either it (non-trivially) avoids one of them, either it fills both of them. If f avoids r_2 then at least two neighbours (in $G_{w\to w'}$) of f fill r_1 (see picture 4.31). If on the contrary f has just one neighbour which fills r_1, then f fills r_2. Under this condition, for any collection F' of regions of w' with $\{r_1, r_2\} \subseteq F'$ we have: $n(f, F') = 2$.

Definition 4.3.33. We set $\sigma(f', F \to F') \stackrel{\text{def}}{=} \sum_{\substack{f\in F \\ f \text{ fills } f'}} \frac{1}{n(f, F')}$. If G is a red graph for w and G' a red graph for w' we write $\sigma(f', G \to G')$ for $\sigma(f', V(G) \to V(G'))$.

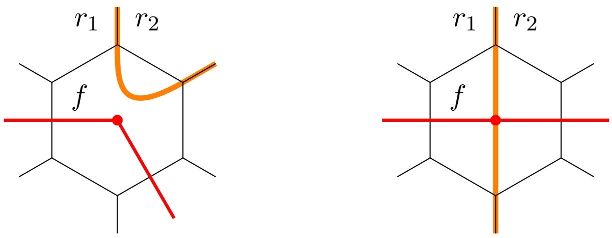


Figure 4.31: On the left f avoids r_2, on the right it fills r_1 and r_2.

4.3.5 Proof of lemma 4.3.16

In this section we use the point of view developed in section 4.3.4 to prove the lemma 4.3.16. We restate it with this new vocabulary:

Lemma 4.3.34. *Let w be an a ∂-connected ε-web which contains no digon and exactly one square. We suppose furthermore that this square touches the unbounded face. Let G be a nice red graphs of w and G' a nice red graph of $w' = w_G$, then there exists $\widetilde{w}$ a nice simplification of w such that:*

A) the ε-webs $(w_G)_{G'}$ and $\widetilde{w}$ are isotopic,

B) the following equality holds:

$$\#V(\widetilde{G}) \geqslant \#V(G) + \#V(G'),$$

where $\widetilde{G}$ denote the red graph $G_{w\to\widetilde{w}}$.

Proof. Because of the condition A, we already know the isotopy class of the web $\widetilde{w}$. To describe it completely, we only need to specify how $\widetilde{w}$ is embedded. For each face f' of w' which is a vertex of G', let us denote $N_{f'}$ a regular neighbourhood of f'. We consider U the complementary of $\bigcup_{f'} N_{f'}$. Provided this is done in a coherent fashion, it's enough to specify how $\widetilde{w}$ looks like in U and in $N_{f'}$ for each face f' of w'.

If f' is a face of w' which is in G', we consider two different cases:

1. the face f' corresponds to a vertex of G' with exterior degree equal to 0,
2. the face f' corresponds to a vertex of G' with exterior degree equal to 2.

These are the only cases to consider since G' is nice.

Let us denote by w'' the ε-web $(w')_{G'}$. We want $\widetilde{w}$ and w'' to be isotopic. So let us look at $w'' \cap U$ and at $w'' \cap N_{f'}$ in the two cases.

Around U, the ε-web w' does not “see” the red graph G', so that $U \cap w'' = U \cap w'_{G'} = U \cap w'$.

If the face f' has exterior degree equal to 0 (case 1), then we have: $N_{f'} \cap w'' = U \cap w'_{G'} = \emptyset$.

If the face f' has exterior degree equal to 2 (case 2), then we have: $N_{f'} \cap w'' = U \cap w'_{G'}$ is a single strands cutting $N_{f'}$ into two parts.

We embed $\widetilde{w}$ such that $U \cap \widetilde{w}$ and $U \cap w''$ are equal and for each face f' corresponding to a vertex of G', $N_{f'} \cap \widetilde{w}$ and $N_{f'} \cap w''$ are isotopic (relatively to the boundary).

We claim that if f' is a vertex with external degree equal to 0 then:

$$\begin{cases} \sigma(f', \widetilde{G} \to G') \geqslant \sigma(f', G \to G') + \frac{1}{2} & \text{if } S \subseteq N_{f'}, \\ \sigma(f', \widetilde{G} \to G') \geqslant \sigma(f', G \to G') + 1 & \text{if } S \nsubseteq N_{f'}, \end{cases} \tag{4.2}$$

where S is the square of w.

The restriction[3] $G_{f'}$ of G' to f' is a graph which satisfies the following conditions:

- it is bi-coloured (because the vertices of G are essential faces of w with respect to w'),
- it is naturally embedded in a disk because $N_{f'}$ is diffeomorphic to a disk,
- the degree of the vertices inside the disks have degree at least three (because the only possible square of w touches the border) and the vertices on the border (these are the one which intersect an other region of w') have degree at least 1.

The regions of $G_{f'}$ and the vertices of one of the two colours of $G_{f'}$ become vertices of $\widetilde{G}$ (see example depicted on figure 4.32). To proves the inequality (4.2), one should carefully count regions of $G_{f'}$. There are two different cases in (4.2) because the remark 4.3.32 do not apply to the square. Hence we can apply the lemma 4.3.36 which proves (4.2).

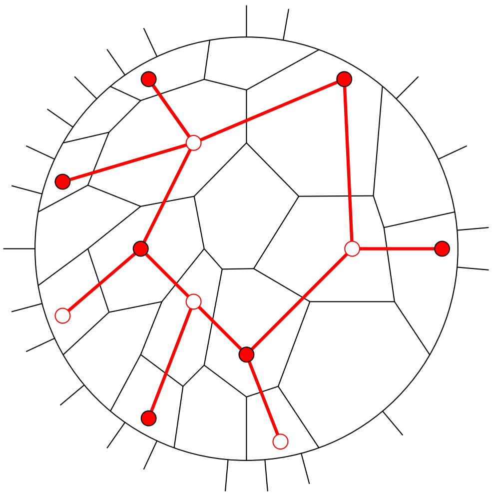


Figure 4.32: Example of the procedure to define $\widetilde{G}$ when the exterior degree of f' is equal to 0.

The only thing remaining to specify is how $\widetilde{w}$ looks in $N_{f'}$ where f' is a vertex of G' of exterior degree equal to 2. Note that we need to embed $\widetilde{w}$ so that it is a nice simplification of w. We claim that it is always possible to find such an embedding so that the inequality (4.2) is satisfied. In this case the graph $G_{f'}$ is in the same situation as before, but is important to notice that the faces of $G_{f'}$ have at least 6 sides (this is a consequence of proposition 4.1.23).

The vertices of $\widetilde{G}$ are the regions of $G_{f'}$ and the vertices of $G_{f'}$ of one of the two colours on one side of the strand and the vertices of $G_{f'}$ of the other colour on the other side of the strand (see figure 4.32 for an example). Hence in order to show that the inequality (4.2) holds, one should carefully count the regions and the vertices of $G_{f'}$, this is done by lemma 4.3.6.

So now we have a simplification $\widetilde{w}$ of w, such that the graph $\widetilde{G} = G_{w\to\widetilde{w}}$ satisfies (4.1.23) for each region f' of G'. The square S of w is in at most one $N_{f'}$ so that if we sum (4.1.23) for all the vertices of f', we obtain:

$$\sum_{f'\in F'} \sigma(f', \widetilde{G} \to G') \geqslant \sum_{f'\in F'} \sigma(f', G \to G') + \#V(G') - \frac{1}{2},$$

[3] We only consider the vertices of G which fill f'.

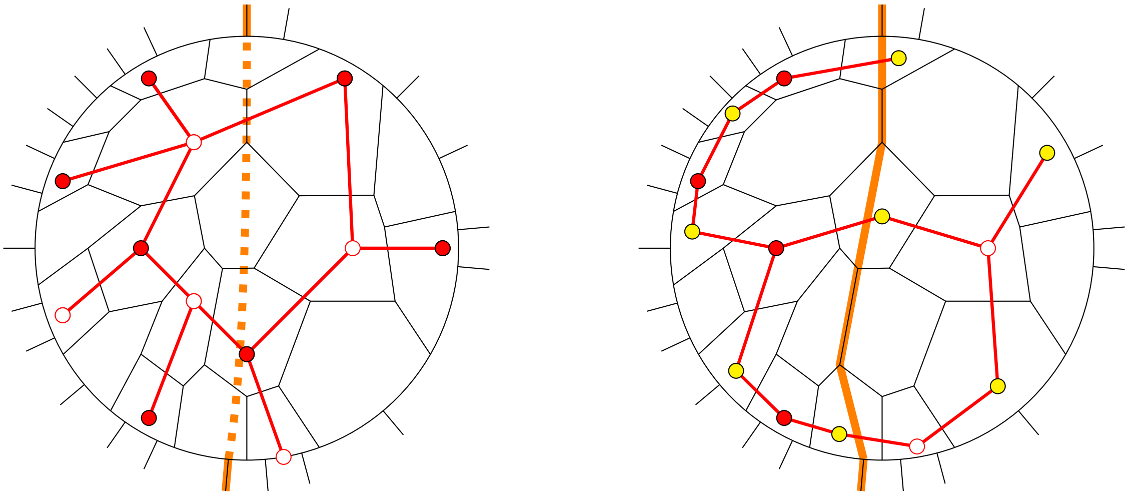


Figure 4.33: Example of the procedure to define $\widetilde{G}$ when the exterior degree of f' is equal to 2.

and using (4.1), we have:

$$\#V(\widetilde{G}) \geqslant \#V(G') + \#V(G) - \frac{1}{2},$$

but $\#V(\widetilde{G})$ being an integer we have $V(\widetilde{G}) \geqslant \#V(G') + \#V(G)$ □

4.3.6 Proof of combinatorial lemmas

This proof is dedicated to the two technical lemmas used in the last section. We first introduce the ad-hoc objects and then state and prove the lemmas.

Definition 4.3.35. A *D-graph* is a graph G embedded in the disk D^2. The set of vertices $V(G)$ is partitioned in two sets: $V^\partial(G)$ contains the vertices lying on ∂D^2, while V^{in} contains the others. The set $F(G)$ of connected components of $D^2 \setminus G$ is partitioned into two sets F^{in} contains the connected component included in $\mathring{D}^2$, while F^∂ contains the others.

A D-graph is said to be *non-elliptic* if:

- every vertex v of V^{in} has degree greater or equal to 3,
- every vertex v of V^∂ has degree greater or equal than 1,
- the faces of F^{in} are of size at least 6.

A *coloured D-graph* is a D-graph G together with:

- a vertex-2-colouring (by green and blue) of the vertices of G (this implies that G is bipartite),
- a subdivision of $\mathcal{D}^2$ into two intervals (we allow one interval to be the empty set and the other one to be the full circle, in this case we say that G is *circled-coloured*): a green one and a blue one (denoted by I_{blue} and I_{green}) when they are real intervals we defines x and y to be the two intersection points of I_{green} and I_{blue} with the convention that when one scans ∂D^2 clockwise, one see x, then I_{green}, then y and finally I_{blue}.

The vertices of V^∂ are supposed neither on x nor on y. The colour of a vertex is not supposed to fit the colour of the interval it lies on. We set V_{green} (resp. V_{blue}) the set of green, (resp. blue) vertices, and $V^\partial_{\text{green}}$, $V^{\text{in}}_{\text{green}}$, V^∂_{blue} and $V^{\text{in}}_{\text{blue}}$ in the obvious way.

If G is a coloured D-graph, and v is a vertex of V^∂ of we set:

$$n(v) = \begin{cases} 2 & \text{if } v \text{ has degree 1 and the colour of } v \text{ fits the colour of the interval,} \\ 1 & \text{else.} \end{cases}$$

If v is a vertex of V^{in}, we set $n(v) = 1$. Note that this definition of n is a translation of the n of the previous section (see remark 4.3.32).

Case with exterior degree equal to 0

Lemma 4.3.36. *Let G be a non-elliptic circled-coloured D-graph (with the circle coloured by a colour c), then:*

$$\#F \geqslant 1 + \sum_{v \in V_c} \frac{1}{n(v)}.$$

Proof. By symmetry, we may suppose that $c = \text{green}$. To show this we consider the graph H obtain by gluing to copies of G along the boundary of D^2 this is naturally embedded in the sphere. We write the Euler characteristic:

$$\#F(H) - \#E(H) + \#V(H) = 1 + \#C(H), \tag{4.3}$$

where $C(H)$ is the set of connected components of H. We have the following equalities:

$$\begin{aligned} \#F(H) &= 2\#F^{\text{in}}(G) + \#F^\partial(G), \\ \#F^\partial(G) &= \#V^\partial(G) + 1 - \#C(H), \\ \#E(H) &= 2\#E(G) = \sum_{v \in V(G)} \deg(v) = 2 \sum_{v \in V_{\text{green}}(G)} \deg(v), \\ \#V(H) &= 2\#V^{\text{in}}(G) + \#V^\partial(G) \end{aligned}$$

So that we can rewrite (4.3):

$$2\#F^{\text{in}}(G) + 2\#F^\partial(G) + 2\#V^{\text{in}}(G) = 2 + 2\#E(G). \tag{4.4}$$

Now we use the what we know about degrees of the vertices:

$$\begin{aligned} \#E(G) &\geqslant \frac{3}{2}\#V^{\text{in}}(G) + \frac{1}{2}\#V^{\partial,1}(G) + 1\#V^{\partial,>1}(G), \\ \#E(G) &\geqslant 3\#V^{\text{in}}_{\text{green}}(G) + \#V^{\partial,1}_{\text{green}}(G) + 2\#V^{\partial,>1}_{\text{green}}(G). \end{aligned}$$

Where $V^{\partial,1}$ (resp. $V^{\partial,>1}$) denotes the subset of V^∂ with degree equal to 1 (resp. with degree strictly bigger than 1). If we sum $\frac{2}{3}$ of the first inequality and $\frac{1}{3}$ of the second one, and inject this in (4.4) we obtain:

$$
\begin{aligned}
\#F(G) + \#V^{\mathrm{in}}(G) &\geqslant 1 + \#E(G) \\
\#F(G) + \#V^{\mathrm{in}}(G) &\geqslant V^{\mathrm{in}}(G) + \frac{1}{3}\#V^{\partial,1}(G) + \frac{2}{3}\#V^{\partial,>1}(G) \\
&\quad + \#V^{\mathrm{in}}_{\mathrm{green}}(G) + \frac{1}{3}\#V^{\partial,1}_{\mathrm{green}}(G) + \frac{2}{3}\#V^{\partial,>1}_{\mathrm{green}}(G) \\
\#F(G) &\geqslant \#V^{\mathrm{in}}_{\mathrm{green}}(G) + \frac{2}{3}\#V^{\partial,1}_{\mathrm{green}}(G) + \frac{4}{3}\#V^{\partial,>1}_{\mathrm{green}}(G) \\
&\quad + \frac{1}{3}\#V^{\partial,1}_{\mathrm{blue}}(G) + \frac{2}{3}\#V^{\partial,>1}_{\mathrm{blue}}(G) \\
&\geqslant \#V^{\mathrm{in}}_{\mathrm{green}}(G) + \frac{1}{2}\#V^{\partial,1}_{\mathrm{green}}(G) + \#V^{\partial,>1}_{\mathrm{green}}(G) \\
&\geqslant \sum_{v \in V_{\mathrm{green}}} \frac{1}{n(v)}.
\end{aligned}
$$

□

Case with exterior degree equal to 2

Lemma 4.3.37. *If G is a non-elliptic D-graph, then all the faces of F are diffeomorphic to disks, and if it is non-empty, then at least one of the following situations happens:*

(1) the set $V^{\partial,>1}$ is non empty,

(2) there exists, two $\cap$'s (see figure 4.34) (if G consists of only one edge, the two $\cap$'s are actually the same one counted two times because it can be seen as a $\cap$ on its two sides),

(3) there exists three λ's or H's (see figure 4.34).

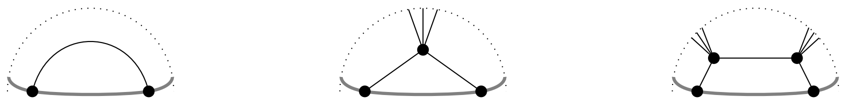


Figure 4.34: From left to right: a $\cap$, a λ and an H. The circle ∂D^2 is thick and grey, the D-graph is thin and black. Note that the vertices inside D^2 may have degree bigger than 3.

Proof. This is the same Euler characteristic-argument that we used in lemma 4.2.20. □

Definition 4.3.38. A cut in a (not circled-) coloured D-graph is a simple oriented path $\gamma : [0,1] \to D$ such that:

- we have $\gamma(0) = x$ and $\gamma(1) = y$, therefore I_{green} is on the left and I_{blue} is on the right[4]. (see figure 4.35),
- for every face f of G, $f \cap \gamma$ is connected,
- the path γ crosses G either transversely at edges joining a green vertex on left and a blue vertex on the right, or at vertices of V^{∂} whose colours do not fit with the intervals they lie on.

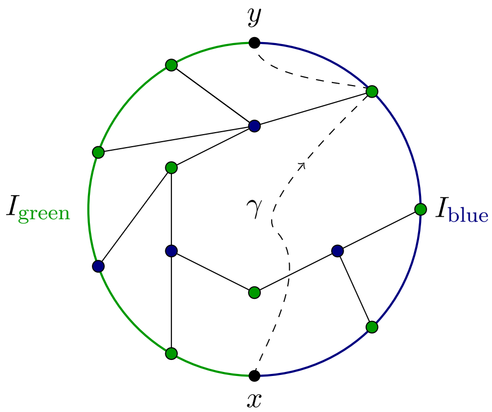


Figure 4.35: A cut in a coloured D-graph (note that G is elliptic).

If γ is a cut we denote by ${}^{l(\gamma)}V(G)$ and ${}^{r(\gamma)}V(G)$ the vertices located on the left (resp. on the right) of γ. (The vertices located on γ are meant to be both on the left and on the right).

Lemma 4.3.39. *Let G be is a non-elliptic (not circled-) coloured D-graph, then there exists a cut γ such that:*

$$\#F(G) \geqslant 1 + \sum_{v \in {}^{l(\gamma)}V_{\text{green}}} \frac{1}{n(v)} + \sum_{v \in {}^{r(\gamma)}V_{\text{blue}}} \frac{1}{n(v)}.$$

Proof. The proof is done by induction on $s(G) \stackrel{\text{def}}{=} 3\#E(G) + 4\#V^{\partial,>1}(G)$. If this quantity is equal to zero then the D-graph is empty, then we choose γ to be any simple arc joining x to y, and the lemma says $1 \geqslant 1$ which is true. We set:

$$C(G,\gamma) \stackrel{\text{def}}{=} \sum_{v \in {}^{l(\gamma)}V_{\text{green}}} \frac{1}{n(v)} + \sum_{v \in {}^{r(\gamma)}V_{\text{blue}}} \frac{1}{n(v)}.$$

It is enough to check the situations (1), (2) and (3) described in lemma 4.3.37.

Situation (1) Let us denote by v a vertex of $V^{\partial,>1}$. There are two cases: the colour of v fits with the colours of the intervals it lies on or not.

If the colours fit, say both are green, we consider G' the same coloured D-graph as G but with v split into v_1, v_2 $\dots v_{\deg(v)}$ all in $V^{\partial,1}(G')$ (see figure 4.36). We have $s(G') = S(G) - 4 < s(G)$ and G' non-elliptic, therefore we can apply the induction hypothesis. We can find a cut γ' with $\#F(G') \geqslant 1 + C(G',\gamma')$. Note that γ' does not cross any v', so that we can lift γ' in the D-graph G. This gives us γ. We have:

$$\begin{aligned} C(G,\gamma) &= C(G',\gamma') + \frac{1}{n(v)} - \sum_{k=1}^{\deg(v)} \frac{1}{n(v_k)} \\ &= C(G',\gamma') + 1 - \frac{\deg(v)}{2} \\ &\geqslant C(G',\gamma'). \end{aligned}$$

On the other hand $\#F(G) = \#F(G')$ so that we have $\#F(G) \geqslant 1 + C(G,\gamma)$.

[4]We use the convention that the left and right side are determined when one scans γ from x to y.

Figure 4.36: Local picture of G (on the left) and G' (on the right) around, when v is green and lies on I_{green}.

If the colours do not fit (say v is blue), we construct G' a coloured D-graph which is similar to G every where but next to v. The vertex v is pushed in D^2 (we denote it by v') and we add a new vertex v'' on ∂D^2 and an edge e joining v' and v''. The coloured D-graph G' is non-elliptic and $s(G') = s(G) - 4 + 3 < s(G)$ so that we can apply the induction hypothesis and find a cut γ' with $\#F(G') \geqslant 1 + C(G', \gamma')$.

Figure 4.37: Local picture of G (on the left) and G' (on the right) around v, when v is blue and lies on I_{green}.

If γ' does not cross e we can lift γ' in G (this gives us γ). We have

$$C(G, \gamma) = C(G', \gamma') + \frac{1}{n(v)} - \frac{1}{n(v')} = C(G', \gamma') + 1 - 1 = C(G', \gamma').$$

On the other hand, we have $F(G) = F(G')$, so that $\#F(G) \geqslant 1 + C(G, \gamma)$.

Consider now the case where γ' crosses e. Then we consider the cut γ of G which is the same as γ far from v, and which around v crosses G in v (see figure 4.38). We have:

$$\begin{aligned}
C(G, \gamma) &= C(G', \gamma') + \frac{1}{n(v)} - \frac{1}{n(v')} - \frac{1}{n(v'')} \\
C(G, \gamma) &= C(G', \gamma') + 1 - 1 - \frac{1}{2} \\
C(G, \gamma) &\geqslant C(G', \gamma').
\end{aligned}$$

But $\#F(G) = \#F(G')$, so that we have $\#F(G) \geqslant 1 + C(G, \gamma)$.

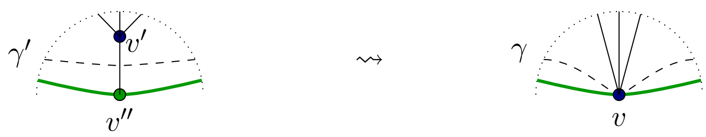


Figure 4.38: How to transform γ' into γ.

Situation (2) We now suppose that G contains two $\cap$'s. Let us denote by v_g (resp. v_b) the green (resp. blue) vertex of the $\cap$ and by e the edge of the cap. There are different possible configurations depending where x and y lies. As there are at least two caps, we may suppose y is far from the $\cap$.

There are 3 different configurations (see figure 4.39):

- the point x is far from the $\cap$,
- the point x is in the $\cap$ and $v_g \in I_{\text{green}}$ and $v_b \in I_{\text{blue}}$,
- the point x is in the $\cap$ and $v_g \in I_{\text{blue}}$ and $v_b \in I_{\text{green}}$,

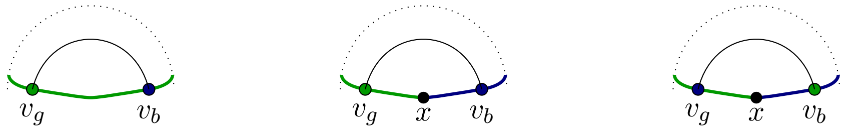


Figure 4.39: The three possible configurations.

We consider G' the coloured D-graph similar to G except that the $\cap$ is removed. The coloured D-graph G' is non-elliptic and $s(G') = s(G) - 3 < s(G)$ so that we can apply the induction hypothesis and find a cut γ' with $\#F(G') \geqslant 1 + C(G', \gamma')$.

Let us suppose first that x is far from the $\cap$, then v_b and v_g both lie either on I_{green} or on I_{blue}. By symmetry we may consider that they both lie on I_{green}. We can lift γ' in G (this gives γ) so that it does not meet the $\cap$. We have:

$$\begin{aligned} C(G,\gamma) &= C(G',\gamma') + \frac{1}{n(v_g)} \\ &= C(G',\gamma') + \frac{1}{2}. \end{aligned}$$

But $\#F(G) = \#F(G') + 1$, hence $\#F(G) \geqslant 1 + C(G, \gamma)$.

Suppose now that the point x is in the $\cap$ and $v_g \in I_{\text{green}}$ and $v_b \in I_{\text{blue}}$. We can lift γ' in G so that it crosses e (see figure 4.40). We have:

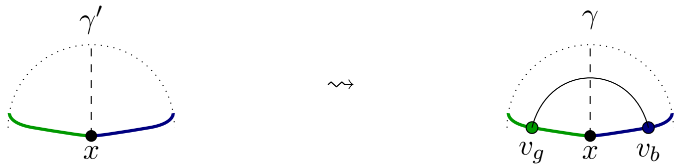


Figure 4.40: How to transform γ' into γ.

$$\begin{aligned} C(G,\gamma) &= C(G',\gamma') + \frac{1}{n(v_g)} + \frac{1}{n(v_b)} \\ &= C(G',\gamma') + \frac{1}{2} + \frac{1}{2} \\ &= C(G',\gamma') + 1. \end{aligned}$$

But $\#F(G) = \#F(G') + 1$, hence $\#F(G) \geqslant 1 + C(G, \gamma)$.

Suppose now that the point x is in the $\cap$ and $v_g \in I_{\text{blue}}$ and $v_b \in I_{\text{green}}$. We can lift γ' in G so that it crosses[5] v_g (see figure 4.41). We have:

$$\begin{aligned} C(G,\gamma) &= C(G',\gamma') + \frac{1}{n(v_g)} \\ &= C(G',\gamma') + 1. \end{aligned}$$

But $\#F(G) = \#F(G') + 1$, hence $\#F(G) \geqslant 1 + C(G, \gamma)$.

[5] We could have chosen to cross v_b.

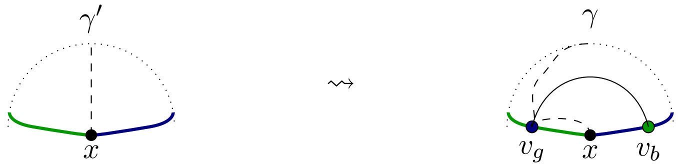


Figure 4.41: How to transform γ' into γ.

Situation (3) We suppose now that there are three λ's or H's. One can suppose that a λ or an H is far from x and from y.

Consider first that there is a λ far from x and y. Let us denote by v_1 and v_2 the two vertices of the λ which belongs to $V^\partial(G)$, by v the vertex of the λ which is in $V^{\text{in}}(G)$ and by e_1 (resp. e_2) the edge joining v to v_1 (resp. v_2). We consider G' the D-graph where the λ is replaced by a single strand: the edges e_1 and e_2 and the vertices v_1 and v_2 are suppressed. The vertex v is moved to ∂D^2 (and renamed v'). This is depicted on figure 4.42. The coloured D-graph G' is non-elliptic and $s(G') < s(G)$ so that we can apply the induction hypothesis and find a cut γ' with $\#F(G') \geqslant 1 + C(G', \gamma')$.

The vertices v_1 and v_2 have the same colour, by symmetry we may suppose that they are both green. It implies that v and v' are both blue.

There are two different configurations:

- the vertices v_1 and v_2 lie on I_{green},
- the vertices v_1 and v_2 lie on I_{blue}.

Figure 4.42: On the center, the two possible configurations for a λ, on the sides, the D-graphs G' obtained from G.

Let us first suppose that the vertices v_1 and v_2 lie on I_{green}. If the cut γ' does not cross v' then we can canonically lift it in G. This gives us γ. We have:

$$\begin{aligned} C(G,\gamma) &= C(G',\gamma') + \frac{1}{n(v_1)} + \frac{1}{n(v_2)} \\ &= C(G',\gamma') + \frac{1}{2} + \frac{1}{2}. \end{aligned}$$

But $\#F(G) = \#F(G') + 1$, hence $\#F(G) \geqslant 1 + C(G, \gamma)$.

If the cut γ' crosses v', we lift γ' in G so that it crosses e_1 and e_2 (see figure 4.41). In

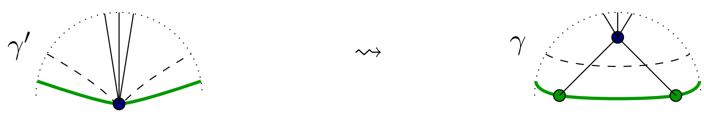


Figure 4.43: How to transform γ' into γ.

this case we have:

$$\begin{aligned} C(G,\gamma) &= C(G',\gamma') + +\frac{1}{n(v)} - \frac{1}{n(v')} + \frac{1}{n(v_1)} + \frac{1}{n(v_2)} \\ &= C(G',\gamma') + 1 - 1 + \frac{1}{2} + \frac{1}{2}. \end{aligned}$$

Hence, $\#F(G) \geqslant 1 + C(G,\gamma)$.

Now suppose that the vertices v_1 and v_2 lie on I_{blue}, this implies that γ' does not meet v', so that we can lift γ' canonically in G, this gives us γ, we have:

$$\begin{aligned} C(G,\gamma) &= C(G',\gamma') + \frac{1}{n(v)} - \frac{1}{n(v')} \\ &= C(G',\gamma') + 1 - 1. \end{aligned}$$

Hence $\#F(G) \geqslant 1 + C(G,\gamma)$.

We finally consider a H far from x and y. We take notation of the figure 4.44 to denote vertices and edges of the H, we consider G' the D-graph where the the H is simplified (see figure 4.44 for details and notation). The coloured D-graph G' is non-elliptic and $s(G') = s(G) - 3 \times 3 + 2 \times 4 < s(G)$ so that we can apply the induction hypothesis and find a cut γ' with $\#F(G') \geqslant 1 + C(G',\gamma')$.

Figure 4.44: How to transform G into G'.

Up to symmetry there is only one configuration, therefore we may suppose that v_1 is green and lies on I_{green}. This implies that v_2 and v_3 are blue and that v_4 is green. Because of the colour condition, the cut γ' does not cross v_4' and may cross v_3'. If it does not cross v_3', one can canonically lift γ' in G' and we have:

$$\begin{aligned} C(G,\gamma) &= C(G',\gamma') + \frac{1}{n(v_1)} + \frac{1}{n(v_4)} - \frac{1}{n(v_4')} \\ &\geqslant C(G',\gamma') + \frac{1}{2} + 1 - 1. \\ &\geqslant C(G',\gamma') + \frac{1}{2}. \end{aligned}$$

But $\#F(G) = \#F(G') + 1$, hence $\#F(G) \geqslant 1 + C(G,\gamma)$.

If the cut γ' crosses v_3', on lift it in $\mathcal{G}$ so that it crosses e_1 and e_2 (see figure 4.45). So

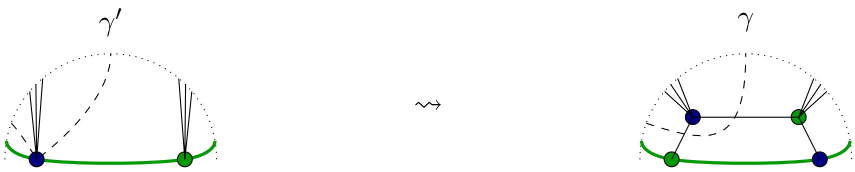


Figure 4.45: How to transform γ' into γ.

that we have:

$$\begin{aligned} C(G,\gamma) &= C(G',\gamma') + \frac{1}{n(v_1)} + \frac{1}{n(v_3)} - \frac{1}{n(v_3')} + \frac{1}{n(v_4)} - \frac{1}{n(v_4')} \\ &\geqslant C(G',\gamma') + \frac{1}{2} + 1 - 1 + 1 - \frac{1}{2}. \\ &\geqslant C(G',\gamma') + 1. \end{aligned}$$

But $\#F(G) = \#F(G') + 1$, hence $\#F(G) \geqslant 1 + C(G,\gamma)$.

Conclusion For all situations, using the induction hypothesis we can construct a cut γ such that: $\#F(G) \geqslant 1 + C(G,\gamma)$. This proves the lemma.

□

Chapter 5

Colouring webs

5.1 Kuperberg bracket via 3-colourings

For the all chapter we fix $\mathcal{S}$ to be the set of colours $\{\text{red}, \text{green}, \text{blue}\}$. It is endowed with a total order[1] on $\mathcal{S}$: red $<$ green $<$ blue. For every element u of $\mathcal{S}$, we denote by $\mathcal{S}_u$ the set $\mathcal{S}_u \setminus \{u\}$, it is endowed with he order induced by the order of $\mathcal{S}$.

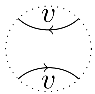
Definition 5.1.1. Let u be a colour of $\mathcal{S}$, a $\mathcal{S}_u$*-coloured cycle* C is either an oriented circle embedded in the plane and coloured with one of the two colours of $\mathcal{S}_u$, or a connected plane oriented 2-valent $\mathcal{S}_u$-edge-coloured graph, such that the vertices are either sinks or sources. By $\mathcal{S}_u$-edge-coloured we mean that there is an application ϕ_C from $E(C)$ the set of edges of C to $\mathcal{S}$ such that two adjacent edges are coloured in a different way. The application ϕ_C is called the *colouring*.

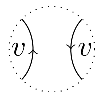
Definition 5.1.2. If C is a $\mathcal{S}_u$-coloured cycle we say that it is *positively oriented* if, when reversing the orientations of the edges which are coloured with the greatest element of $\mathcal{S}_u$, the result is a counterclockwise oriented cycle. When a $\mathcal{S}_u$-coloured cycle is not positively oriented it is *negatively oriented*.

Definition 5.1.3. Let u be an element of $\mathcal{S}$, a $\mathcal{S}_u$*-coloured configuration* D is a finite disjoint union of $\mathcal{S}_u$-coloured cycles $(C_i)_{i\in I}$.

Definition 5.1.4. The *degree* $d(D)$ of an $\mathcal{S}_u$-coloured configuration D is the algebraic number of oriented cycles, *i.e.* the number of positively oriented cycles minus the number of the negatively oriented cycles.

Lemma 5.1.5. *If u is a colour of $\mathcal{S}$, denote by v the smallest colour of $\mathcal{S}_u$. Suppose that D_1 and D_2 are two $\mathcal{S}_u$-coloured configurations which are the same except in a small ball where:*

$D_1 =$ 

and $D_2 =$ 

.

Then we have $d(D_2) - d(D_1) = 1$.

Proof. To show this lemma, it's enough to consider only the connected components of D_1 and D_2 which meet the ball where D_1 and D_2 are different. We can as well suppose that these connected components are coloured with only one colour (which is v, the lowest colour of $\mathcal{S}_u$). Then there are different situations to check, it's enough to inspect the case

[1]This is the frequency order.

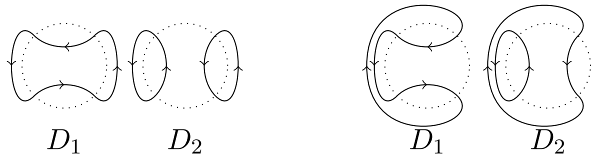


Figure 5.1: There are two different situtations. The dotted circles represent the border of the ball where D_1 and D_2 are different. It's clear that $d(D_2) - d(D_1) = 1$ for both cases.

where D_1 has just one connected component (for otherwise D_2 has only one, and reversing the orientations we are back in the situation where D_1 has only one connected component). There are essentially two possibilities and they are depicted in figure 5.1. □

Definition 5.1.6. If w is a web and c is an $\mathcal{S}$-edge-colouring of w (we mean a function from the set of edges (and circles) of w to $\mathcal{S}$ such that if two edges are adjacent, they have different colours), we denote by w_c the web together with this colouring. If u is a colour of $\mathcal{S}$, we define the *$\mathcal{S}_u$-configuration of* w_c denoted by $D_u(w_c)$, the $\mathcal{S}_u$-coloured configuration obtained from w_c by deleting all the edges (and circles) with colour u. We denote by $\mathrm{col}(w)$ the set of all $\mathcal{S}$-edge-colourings of w.

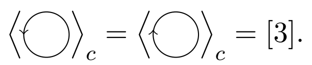
Definition 5.1.7. If w_c is a $\mathcal{S}$-coloured web, we define $d_t(w_c)$ the *total degree of* w_c to be the sum of the $D_u(w_c)$ when u runs over the colours of $\mathcal{S}$. We define $\langle w \rangle_c$ *the coloured Kuperberg bracket of* w to be the Laurent polynomial defined by the following formula:

$$\langle w \rangle_c = \sum_{c \in \mathrm{col}(w)} \prod_{u \in \mathcal{S}} q^{d(D_u(w_c))} = \sum_{c \in \mathrm{col}(w)} q^{d_t(w_c)}.$$

Theorem 5.1.8. *If w is a web then $\langle w \rangle = \langle w \rangle_c$, where $\langle \cdot \rangle$ denote the Kuperberg bracket(see definition 1.1.13).*

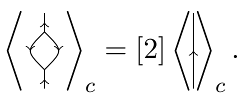
This result is the quantum counterpart of the fact that the evaluation at 1 of the Kuperberg bracket gives the number of colourings of a web (see remark 1.1.14). It's enough to show that $\langle \cdot \rangle_c$ satisfies the relations that define $\langle \cdot \rangle$. We separate the proof in three different lemmas corresponding to the three relations which are known to be enough to completely define the Kuperberg bracket.

Lemma 5.1.9. *The coloured Kuperberg bracket is multiplicative for connected components and we have:*



Proof. The first point is obvious from the definitions. The second point is just a computation. Suppose that the web w is the circle counterclockwise oriented, then there are three different $\mathcal{S}$-edge-colourings for w: the circle is either red, green or blue. The details of the computation is given in the table 5.1. And it's clear from this computations that $\langle w \rangle_c = [3]$. The computation is similar for a clockwise oriented circle. □

Lemma 5.1.10. *The coloured Kuperberg bracket satisfies the following formula:*



	w_{red}	w_{green}	w_{blue}
$D_{\text{red}}(\cdot)$	0	-1	-1
$D_{\text{green}}(\cdot)$	1	0	-1
$D_{\text{blue}}(\cdot)$	1	1	0
$d_t(\cdot)$	2	0	-2

Table 5.1: Degrees of the colourings of the counterclockwisely oriented circle.

Proof. We denote by w the web with the digon and w' the web with this digon replaced by a single strand. The only thing to do is to exhibit two functions ϕ_+ and ϕ_- from $\text{col}(w')$ to $\text{col}(w)$ which are injective, whose images constitute a partition of $\text{col}(w)$ and such that ϕ_+ increases the total degree of colourings by one and ϕ_- decreases the total degree of colourings by one. The functions ϕ_+ and ϕ_- are explicitly described in table 5.2. The

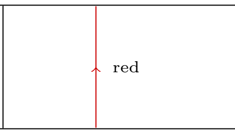
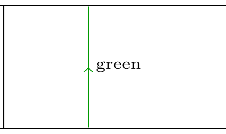
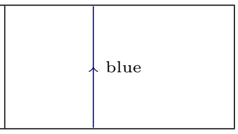
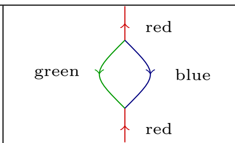
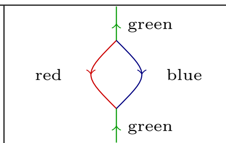
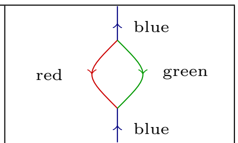
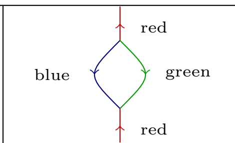
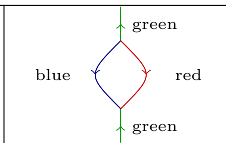
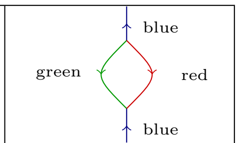
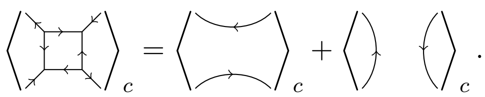
w'_c			
$w_{\phi_+(c)}$			
Δ^+_{red}	$+1$	0	0
Δ^+_{green}	0	$+1$	0
Δ^+_{blue}	0	0	$+1$
$w_{\phi_-(c)}$			
Δ^-_{red}	-1	0	0
Δ^-_{green}	0	-1	0
Δ^-_{blue}	0	0	-1

Table 5.2: The functions ϕ_+ and ϕ_-. The drawn part is the only non-trivial part, where Δ^+_* stands for $d(D_*(w_{\phi_+(c)})) - d(D_*(w'_c))$ and Δ^-_* for $d(D_*(w_{\phi_-(c)})) - d(D_*(w'_c))$.

injectivity of the maps ϕ_+ and ϕ_- is obvious from their definitions, the fact that their images form a partition of $\text{col}(w)$ as well. The only thing to check is the degree conditions. It's an easy computation, the details are given in table 5.2. □

Lemma 5.1.11. *The coloured Kuperberg bracket satisfies the following formula:*



Proof. We denote by w the web with a square figure, w' the web where the square is replaced by two horizontal strands and w'' the web where the square is replaced by two vertical strands. It's enough to describe two injective maps ϕ' and ϕ'' from respectively col(w') and col(w'') to col(w) which preserve the total degree of the colourings and such that their images form a partition of col(w). We describe them in the tables 5.3, 5.4, 5.5 and 5.6 depending on if the two strands have the same colour or not. In tables 5.5 and 5.6, the missing cases are obtained by a 180 degree rotation.

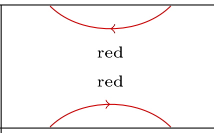
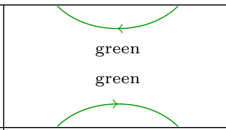
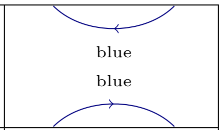
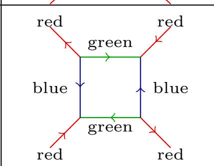
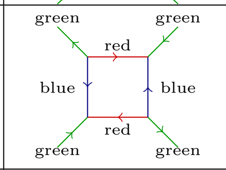
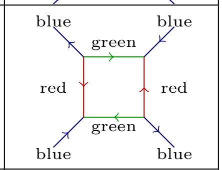
w'_c			
$w_{\phi'(c)}$			
Δ'_{red}	-1	$\underline{+1}$	0
Δ'_{green}	$\underline{+1}$	-1	-1
Δ'_{blue}	0	0	$\underline{+1}$

Table 5.3: Description of the map ϕ' for the colourings which give the same colour to the two horizontal strands of the web w', where Δ'_* stands for $d(D_*(w_{\phi'(c)})) - d(D_*(w'_c))$. To compute the underlined values we use lemma 5.1.5

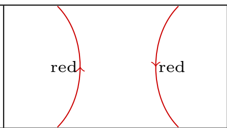
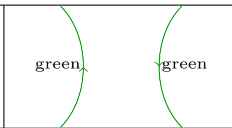
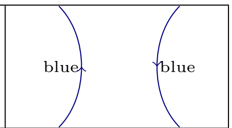
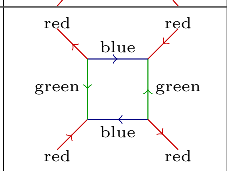
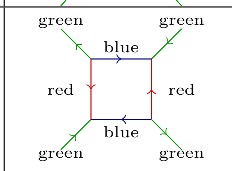
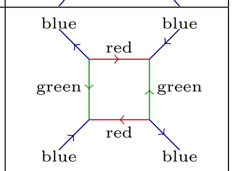
w''_c			
$w_{\phi''(c)}$			
Δ''_{red}	$+1$	$\underline{-1}$	0
Δ''_{green}	$\underline{-1}$	$+1$	$+1$
Δ''_{blue}	0	0	$\underline{-1}$

Table 5.4: Description of the map ϕ'' for the colourings which give the same colour to the two vertical strands of the web w'', where Δ''_* stands for $d(D_*(w_{\phi''(c)})) - d(D_*(w''_c))$. To compute the underlined values we use lemma 5.1.5

The injectivity of the maps ϕ' and ϕ'' is clear. The fact that their images form a partition of col(w) is as well straightforward. For every colouring the sum of the three values of Δ' or Δ'' is equal to zero, and this gives the homogeneity of ϕ' and ϕ''. $\square$

Remark 5.1.12. One could try to use this new approach of the Kuperberg bracket to have a new approach to the $\mathfrak{sl}_3$-homology for links: one can associate to a web the graded vector space with basis the colouring. However, the way to define the differential corresponding to the crossing is not clear. The main obstruction comes from the fact that the degree of a morphism can not be understood locally. A better understanding on how the orientations and the colours interact on the degrees may help. One can define a cheap version of the

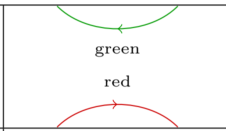
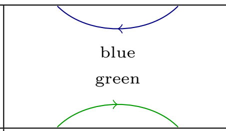
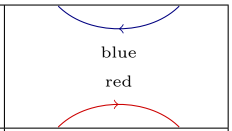
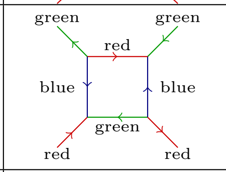
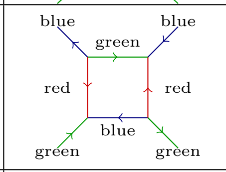
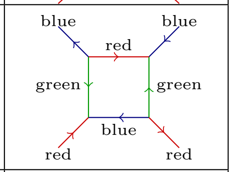
w'_c			
$w_{\phi'(c)}$			
Δ''_{red}	0	0	0
Δ''_{green}	0	0	0
Δ''_{blue}	0	0	0

Table 5.5: Description of the map ϕ' for the colourings which give different colours to the two horizontal strands of the web w', where Δ'_* stands for $d(D_*(w_{\phi'(c)})) - d(D_*(w'_c))$.

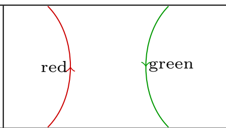
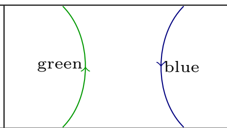
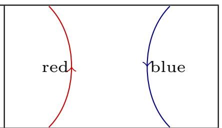
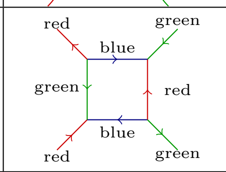
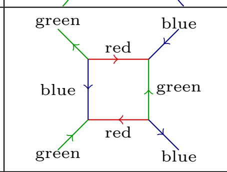
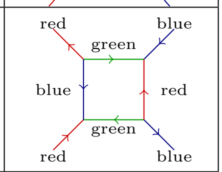
w''_c			
$w_{\phi''(c)}$			
Δ''_{red}	0	0	0
Δ''_{green}	0	0	0
Δ''_{blue}	0	0	0

Table 5.6: Description of the map ϕ'' for the colourings which give different colours to the two horizontal strands of the web w'', where Δ''_* stands for $d(D_*(w_{\phi''(c)})) - d(D_*(w''_c))$.

$\mathfrak{sl}_3$-homology which is not quantified and hence categorify only the number of colour of a web. It turns out that this invariant does not give any information on the links.

5.2 Combinatorics of colourings

Given a web w, one can ask how to go from one colouring to another one. One can of course let $\mathfrak{S}_3$ act on the colourings via interchanging the colours, and for example when exchanging red and blue, we find a colouring with opposite degree. However this doesn't change the structure of the colouring. In this section we propose a semi-local move on colourings and we show that this move is enough to connect any two colourings of w (see theorem 5.2.13).

Definition 5.2.1. Let u be a colour of $\mathcal{S}$. Let w_c be a closed coloured web, we say that a simple cycle[2] C of edges of w_c is $\mathcal{S}_u$*-bi-coloured*, if the edges of C have colours in $\mathcal{S}_u$. When we don't want to focus on the colours we say that it's *bi-coloured*. A $\mathcal{S}_u$-bi-coloured cycle is *positively* or *negatively oriented* if its image in $D_u(w_c)$ is respectively positively or

[2]The term "cycle" includes vertex-less loops.

negatively oriented. If C is a $\mathcal{S}_u$-bi-coloured cycle, then we define the colouring $c' = \tau_C(c)$ to be the colouring of w which is the same as c on all edges which doesn't belong to C and which exchanges the colours of $\mathcal{S}_u$ on the edges in C. It's immediate that the cycle C remains $\mathcal{S}_u$-coloured in $w_{\tau_C(c)}$ that we have $c = \tau_C(\tau_C(c))$.

Proposition 5.2.2. *Let w_c be a closed coloured web, u a colour of $\mathcal{S}$ different from* green *and C a $\mathcal{S}_u$-bi-coloured cycle in w_c. Then we have*

$$d_t(w_{\tau_C(c)}) = \begin{cases} d_t(w_c) - 2 & \text{if } C \text{ is positively oriented,} \\ d_t(w_c) + 2 & \text{if } C \text{ is positively oriented.} \end{cases}$$

Before proving this result we need to introduce a few notions:

Definition 5.2.3. Let u be a colour of $\mathcal{S}$, a *$\mathcal{S}_u$-coloured arc* C is a connected oriented $\mathcal{S}_u$-edge-coloured graph embedded in $\mathbb{R} \times [0,1]$ which satisfies the following conditions:

- exactly two vertices of C have valence 1, all the other have valence 2,
- the 1-valent vertices are located in $\mathbb{R} \times \{0,1\}$, and they are the only intersection of C with $\mathbb{R} \times \{0,1\}$,
- the vertices are either sources or sinks,
- two adjacent edges have different colours.

As before the *colouring* of C is the function from $E(C)$ to $\mathcal{S}_u$.

Definition 5.2.4. Let u be a colour of $\mathcal{S}$ and C a $\mathcal{S}_u$-coloured arc. We say that C is 0-*oriented* if one of its 1-valent vertices is on $\mathbb{R} \times \{0\}$ and the other one on $\mathbb{R} \times \{1\}$. Now let us reverse the orientations of the edges coloured by the highest colour of $\mathcal{S}_u$, the arc C is now coherently oriented and it has a tail and a head which have coordinates (x_t, y_t) and (x_h, y_h). It is *positively oriented* if we are in one of the following situations:

- the ordinates are equal to 0 and the abscissae satisfy $x_h < x_t$,
- the ordinates are equal to 1 and the abscissae satisfy $x_t < x_h$.

A $\mathcal{S}_u$-coloured arc which is neither 0-oriented nor positively oriented is *negatively oriented.*

Definition 5.2.5. A *$\mathcal{S}_u$-path configuration* is a disjoint union of some $\mathcal{S}_u$-coloured arcs and $\mathcal{S}_u$-coloured cycles which all lie in $\mathbb{R} \times [0,1]$.

Two $\mathcal{S}_u$-path configurations can be composed by glued whenever they are compatible, by stacking them and resizing.

Definition 5.2.6. The *degree* $d(D)$ of a $\mathcal{S}_u$-path configuration D is the number given by the formula:

$$d(D) = p_c - n_c + \frac{1}{2}(p_a - n_a),$$

where p_c (resp. n_c) stands for the number of positively (resp. negatively) oriented $\mathcal{S}_u$-coloured cycles and p_a (resp. n_a) the number of positively (resp. negatively) oriented $\mathcal{S}_u$-coloured arcs.

Lemma 5.2.7. *If D_b and D_t are two $\mathcal{S}_u$-path configurations, so that the composition $D = D_b \circ D_t$ (we mean here that D_b is under D_t) is defined then we have $d(D) = d(D_t) + d(D_b)$.*

Proof. It comes from the fact that the degree we defined is actually a normalised version of the total curvature of an oriented curve, which is equal to the integral of the signed curvature along the curve. And the Chasles relation gives the result. □

Lemma 5.2.8. *The lemma 5.1.5 holds as well for $\mathcal{S}_u$-path configurations.*

Proof. This is clear from the previous lemma. □

Proof of proposition 5.2.2. If C is a vertex-less loop the statement is obvious so we consider the other cases. The colouring $\tau_C(c)$ is denoted by c'. As the two situations are symmetric we may assume that $u = \text{blue}$. The $\mathcal{S}_u$-coloured configurations $D_{\text{blue}}(w_c)$ and $D_{\text{blue}}(w_{c'})$ are the same except for the cycle corresponding to C which is coloured in the opposite manner and hence oriented in the opposite manner and we then have: $d(D_{\text{blue}}(w_c)) = d(D_{\text{blue}}(w_{c'})) \pm 2$ (depending on how C is oriented). We now look at $D_* = D_*(w_c)$ and $D'_* = D_*(w_{c'})$ for $* = \text{green}$ and red. We can perform an isotopy so that the cycle C has all but one of its edges horizontal, and the non-horizontal one, is over the horizontal ones (see figure 5.2). We now see w as a composition of three web tangles: $w = w^3 \circ w^2 \circ w^1$, where w^2 contains

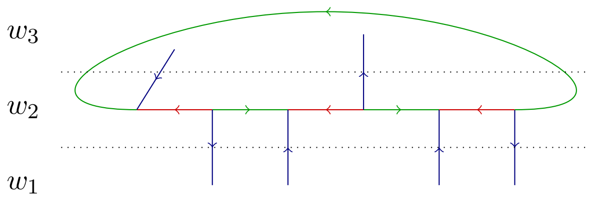


Figure 5.2: We decompose w into three web tangles: w_1, w_2 and w_3

only the horizontal edges of C, two parts of the non-horizontal edge of C, the half edges of w which are touching C (they all have colour blue in w_c), and vertical arcs far from C. It's clear from our hypotheses that $w^1_c = w^1_{c'}$ and w^3_c and $w^3_{c'}$ differ just by one arc which is red in one and green in the other one, but as blue is bigger than this two colours, we have $d(D_{\text{green}}(w^3_c)) = d(D_{\text{green}}(w^3_{c'})) \pm 1$ and $d(D_{\text{red}}(w^3_{c'})) = d(D_{\text{red}}(w^3_c)) \pm 1$, where the $\pm$ signs is the same in both cases. On the other hand we have: $d(D_{\text{red}}(w^2_c)) = d(D_{\text{green}}(w^2_{c'}))$ and $d(D_{\text{green}}(w^2_c)) = d(D_{\text{red}}(w^2_{c'}))$. And summing all this equalities together and using the lemma 5.2.7 we conclude that:

$$d(D_{\text{red}}(w_c)) + d(D_{\text{green}}(w_c)) = d(D_{\text{red}}(w_{c'})) + d(D_{\text{green}}(w_{c'})).$$

This is enough to conclude that $d_t(w_c) = d_t(w_{c'}) \pm 2$, depending on how C is oriented. □

We now define two equivalence relations on $\text{col}(w)$ and the rest of this section will be devoted two show that for a given web w all the colourings are equivalent (see theorem 5.2.13).

Definition 5.2.9. Let w be a closed web, two colourings c and c' are said to be weakly τ-close if one can find a bi-coloured cycle C in w_c so that $c' = \tau_C(c)$. They are said to be strongly τ-close if one can find a $\mathcal{S}_{\text{blue}}$- or a $\mathcal{S}_{\text{red}}$-bi-coloured cycle C in w_c so that $c' = \tau_C(c)$. The weak (resp. strong) τ-equivalence relation is generated by the pairs (c, c') for c and c' (resp. strongly) τ-close from each other.

Remark 5.2.10. It's clear from the definition that if c and c' are weakly τ-equivalent, then they are strongly τ-equivalent. The notion of strongly τ-closeness, makes sense in view of proposition 5.2.2. Before proving theorem 5.2.13, we show that the two different notions of τ-equivalence are the same (see lemma 5.2.12).

Lemma 5.2.11. *Consider a coloured web w and two edges e_1 and e_2 as in figure 5.3, then if e_1 and e_2 are coloured differently, then they do not belong to the same bi-coloured cycle.*

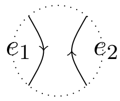


Figure 5.3: The edges e_1 and e_2 with their orientations.

Proof. If the two edges have different colours, their orientation are not compatible to fit in the same bi-coloured cycle. □

Lemma 5.2.12. *If w is a closed web, and c and c' are weakly τ-equivalent colourings of w, then c and c' are strongly τ-equivalent.*

Proof. It's enough to show that if two colourings c and c' are weakly τ-close then they are strongly τ-equivalent. As the other cases are straightforward let us suppose that $c' = \tau_C(c)$ with C a $\mathcal{S}_{\text{green}}$-bi-coloured cycle. Consider (C'_i) the collection of all the $\mathcal{S}_{\text{blue}}$-bi-coloured cycles in w_c which intersect C (it should be at some red edges). The C'_i's are all disjoint so that we can define $c_1 = \tau_{(C'_i)}(c)$ the colouring similar to c except that on all C'_i's, the colours red and green are exchanged. It's clear that c and c_1 are strongly τ-equivalent. In w_{c_1}, C is now $\mathcal{S}_{\text{red}}$-bi-coloured, so that we can define $c_2 = \tau_C(c_1)$. Now we consider (C''_i) the collection of all the $\mathcal{S}_{\text{blue}}$-bi-coloured cycles in w_{c_2} which intersect C (it should be at some green edges). We define $c_3 = \tau_{(C''_i)}(c_2)$ as we defined c_1. It's clear that c_3 is strongly τ-equivalent to c_2 and hence to c_1. It's easy to check that $c_3 = c'$ because the collection of edges of the C'_i's disjoint from C and the the collection of edges of the C''_i's disjoint from C are equal. □

Because of lemma 5.2.12 we speak of τ-equivalence instead of strong or weak τ-equivalences.

Theorem 5.2.13. *Let w be a closed web then all colourings of w are τ-equivalent.*

Proof. We show this by induction on the number of vertices, using the fact that any closed web contains a circle, a digon or a square. Let w be a web and c_0 and c_1 be two colourings of w.

If w contains a circle C, then we consider w' the web w with this circle removed. c_0 and c_1 induce colourings c'_0 and c'_1 on w'. By induction we know that c'_0 and c'_1 are τ-equivalent. All the cycles of w' needed to go via τ-moves from c'_0 to c'_1 can be lifted in w. Hence we obtained a colouring $\tilde{c}_1$, which is τ-equivalent to c_0 and equal to c_1 everywhere but maybe on the circle, so if necessary we perform τ_C on $\tilde{c}_1$ where C is seen as a bi-coloured circle and we obtain c_1 and this show that c_0 and c_1 are τ-equivalent.

Suppose now that w contains a digon. We do the same thing, we consider the web w' where the digon figure is replaced by a single strand, and we can play the same game because for any colouring the two strands of the digon form a bicoloured cycle.

Now if w contain a square, thanks to lemma 5.2.11, we may suppose (up to a single τ-move) that c_0 has the same colour on all the four strands touching the square. We consider the web w' corresponding to one smoothing of the square which is so that c_0 and c_1 induces naturally colourings on w'. We denote them by c'_0 and c'_1. Let c be a colouring of w'. If the two strands of w' which replace the square are coloured in different manners, there is a canonical way to construct a colouring of w and any bi-coloured cycle of w'

can be lifted up in w. If the two strands have the same colour, there are two different colourings of w which can induce c' on w', but these two are obviously τ-close. And if C is a bi-coloured cycle in $w'_{c'}$ then it can be lifted up in w to a bi-coloured cycle endowed with one of this two colouring. So we can play the same game as before: thanks to a path of two by two τ-close colouring from c'_0 to c'_1, we can construct a (maybe longer) path of two by two τ-close colourings from c_0 to c_1. And this concludes. □

One can extend the τ-equivalence to colouring of ε-webs, by allowing bi-coloured arcs in definition 5.2.9 where arcs are supposed to begin and to end with a 1-vertex, we have then following easy corollary:

Corollary 5.2.14. *Let $(\varepsilon, c_\varepsilon)$ be a coloured sequence of signs. Let w be an ε-web and c_1 and c_2 be two colourings of w. Then they are τ-equivalent.*

Proof. We consider the closed web $u = \overline{w}w$, the colouring c_1 and c_2 induced some colourings c'_1 and c'_2 on u by taking their mirror images on $\overline{w}$. Theorem 5.2.13 tells us that c'_1 and c'_2 are τ-equivalent, now following a path of two by two τ-close colourings we see that the restriction of this colouring are either τ-close either τ equivalent via a sequence of bi-coloured arcs. This show that c_1 and c_2 are τ-equivalent. □

One may ask if the theorem 5.2.13 remains true when we consider trivalent $\mathcal{S}$-edge-colourable graphs which are not webs. We show in the following that when one removes the planarity condition or the bipartite condition, the result fails.

Example 5.2.15. The dodecahedral graph G does not satisfy the theorem 5.2.13.

Proof. Consider the colouring c given by figure 5.4. We claim (see figure 5.5) that for every colour u in $\mathcal{S}$, it contains only one $\mathcal{S}_u$ bi-coloured cycle C_u, so that $\tau_{C_u}(c)$ is the same colouring as c but with the two colours of $\mathcal{S}_u$ exchanged, to that the structure of c is not essentially changed.

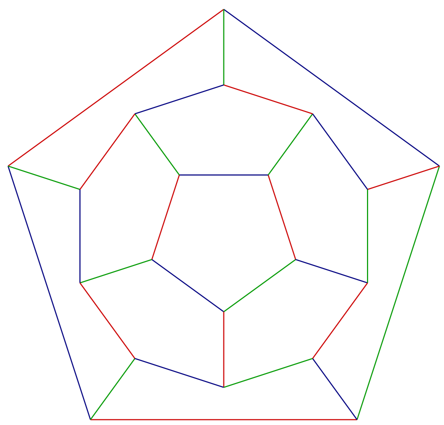


Figure 5.4: The colouring c of the dodecahedral graph

On the other hand, there exist some colourings of G which are essentially different, for example one can perform a $\frac{2\pi}{5}$ rotation of the colouring c to obtain a colouring c', therefor c' and c are not τ-equivalent. □

Example 5.2.16. The utility graph (also called $K_{3,3}$) does not satisfied the theorem 5.2.13.

One can apply the same proof as before, figure 5.6. illustrates it.

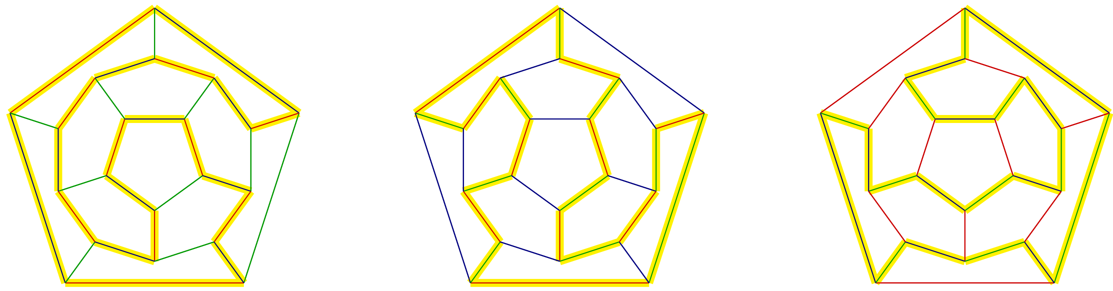


Figure 5.5: The three bi-coloured cycle in G_c.

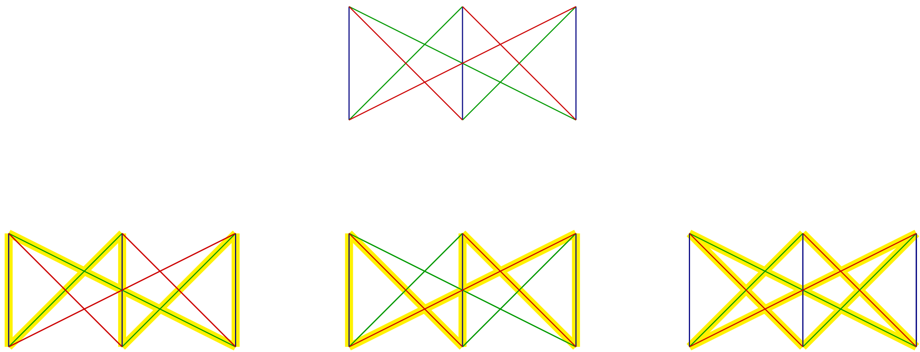


Figure 5.6: A colouring of the graph $K_{3,3}$, and the three bi-coloured cycles for this colouring.

5.2.1 Application to the computation of partial traces

We use the notations of chapter 2. Let us consider a w an ε-web and f a (w,w)-foam. One can "half-close" this foam on itself by gluing the top of this foam to the bottom along the ε-web w. The boundary of the resulting foam $\hat{f}$ is diffeomorphic to a collection of $l(\varepsilon)$ disjoint circles. If we apply the functor $\mathcal{F}$ to $\hat{f}$ we obtain an element of $\mathcal{F}(l(\varepsilon)\text{ circles}) = R[X_1,\dots,X_{l(\varepsilon)}]/(X_i^3=0) \stackrel{\text{def}}{=} A_{l(\varepsilon)}$.

The all procedure gives us a graded R-linear map from $\hom_{K^\varepsilon}(\mathcal{F}(w),\mathcal{F}(w))$ to the R-module $A_{l(\varepsilon)}$. It looks like braid closure or like mapping cones and we will therefore denote it by $\mathrm{Tr}(\cdot)$ and call it *trace*. As it is natural for traces we have the commutative property:

Lemma 5.2.17. *Let w be an ε-web and f and g be two (w,w)-foams, then we have* $\mathrm{Tr}(fg)=\mathrm{Tr}(gf)$, *more precisely the foams $\widehat{fg}$ and $\widehat{gf}$ are homotopic.*

Proof. This is the same proof as for every topological trace, g can travel around f along the circle, the homotopy is depicted on figure 5.7. □

Using the colourings, one can actually compute the trace of 1_{P_w} *i.e.* of the (w,w)-foam $w\times I$. The rest of the section is devoted to exhibit this computation.

Definition 5.2.18. Let w be a coloured ε-web, and let us consider a vertex v of w. We say that v is a *positive* or *negative vertex* according to how the colourings of its adjacent edges look like (see figure 5.8).

Definition 5.2.19. Let $\varepsilon=(\varepsilon^i)_{i\in[1,n]}$ be a sequence of signs, a *colouring of* ε is a function from $[1,n]$ to $\mathcal{S}$. The set of colourings of ε is denoted by $\mathrm{col}(\varepsilon)$. A *coloured sequence of*

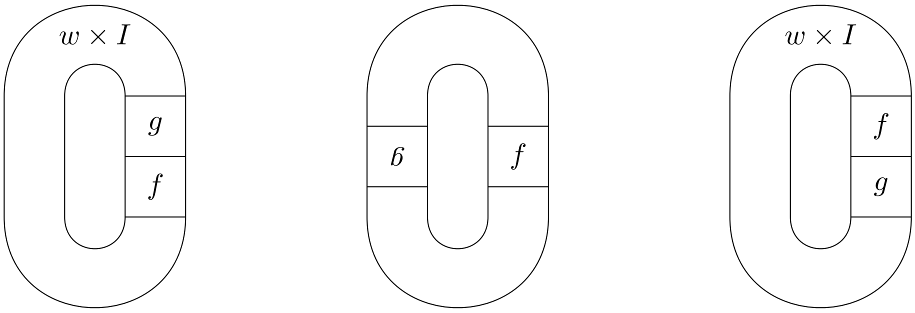


Figure 5.7: On the left $\widehat{fg}$, on the right $\widehat{gf}$.

Figure 5.8: On the left the two local pictures of a positive vertex, on the right, the two local pictures of a negative vertex. Note that positive vertices are mirror images from negative ones and vice-versa.

signs is a sequence of signs together with a colouring. If w is an ε-web, we say that *a colouring c_ε of ε and a colouring c_w of w match*, if for every $i \in \{1, l(\varepsilon)\}$ the colour of the strand ending at ε^i given by c_w is equal to $c_\varepsilon(i)$. For a colouring c of w we defined *the colouring of ε induced by c* to be the only colouring of ε which matches c, we denoted by $c_{|\varepsilon}$.

Lemma 5.2.20. *Let $(\varepsilon, c_\varepsilon)$ a coloured sequence of signs, and let w an ε-web and c_1 and c_2 two colourings of w which match c_ε, then the numbers of negative vertices of w_{c_1} and w_{c_2} have the same parity.*

Proof. We will first show the result holds for a closed web. In this case the matching condition is empty. Thanks to theorem 5.2.13 one only need to show that the parity of negative vertices is preserved by τ-equivalence. This is enough to show it for two τ-close colourings. But if c is a colouring and C is a bi-coloured cycle in C then the vertices of w_c and $w_{tau_C(c)}$ look the same except along C where every positive vertex is changed to a negative one and vice-versa. But as a web is a bipartite graph, the cycle has an even number of vertices, so that in the end the numbers of negative vertices of c and $\tau_C(c)$ have the same parity. Now consider the closed web $w' = \overline{w}w$, and two colourings of w': c_1' is given by c_1 on w and by the mirror of c_1 on $\overline{w}$, and c_2' is given by c_2 on w and the mirror image of c_1 on $\overline{w}$. Note that it is essential that c_1 and c_2 both match c_ε for this construction. For any colouring c of w the numbers of positive and negative vertices of w_c and of $(\overline{w}, \overline{c})$ are exchanged (see of figure 5.8). For c a colouring of a web v denote by $n_-^v(c)$ (resp. $n_+^v(c)$) the number of negative (resp. positive) vertexes of v_c. All this information together gives:

$$\begin{aligned}
n_-^{w'}(c_1') &= n_-^w(c_1) + n_+^w(c_1), \\
n_-^{w'}(c_2') &= n_-^w(c_1) + n_+^w(c_1), \\
n_-^{w'}(c_1') &\equiv n_-^{w'}(c_2') && \pmod 2, \\
n_-^w(c_1) + n_+^w(c_1) &\equiv n_-^w(c_2) + n_+^w(c_1) && \pmod 2, \\
n_-^w(c_1) &\equiv n_-^w(c_2) && \pmod 2.
\end{aligned}$$

□

For a given ε-web w and a colouring c_ε of ε such that their exists a colouring of w which match c_ε, the previous lemma allows us to define $(-1)^{n_-^w(c_\varepsilon)}$ to be equal to $(-1)^{n_-^w(c_w)}$ for any colourings of w which matches c_ε.

Notation 5.2.21. Let t be the function from $\mathcal{S}$ to $\{-1,0,1\}$ defined by $t(\text{red}) = -1$, $t(\text{green}) = 0$ and $t(\text{blue}) = 1$. If (ε, c) is a coloured sequence of signs we denote by $P(\varepsilon, c)$ the monomial of $A_{l(\varepsilon)}$ given by the formula:

$$P(\varepsilon, c) = \prod_{i=1}^{l(\varepsilon)} X_i^{1+\varepsilon^i t(c(i))}.$$

Theorem 5.2.22. *Let w be an ε, then:*

$$\begin{aligned}
\mathrm{Tr}(\mathrm{id}_w) &= (-1)^{e(w)} \sum_{c\in \mathrm{col} w} (-1)^{n_-^w(c)} P(\varepsilon, c_{|\varepsilon}) \\
&= (-1)^{e(w)} \sum_{c_\varepsilon \in \mathrm{col}(\varepsilon)} (-1)^{n_-(c)} P(\varepsilon, c) \cdot \#\{c \in \mathrm{col}(w) \text{ such that } c \text{ matches } c_\varepsilon\},
\end{aligned}$$

where id_w is the (w,w)-foam $I \times w$ and $e(w)$ stands for the number of edges of w.

Remark 5.2.23. The case $\varepsilon = \emptyset$ gives back[3] that the (non-graded) dimension of P_w is equal to the Kuperberg bracket evaluated in 1. In fact, what is computed is the foam-evaluation of $w \times \mathbb{S}^1$, which is known to be the trace, in the classical sense, of the identity of $\mathcal{F}(w)$ *i.e.* the dimension of $\mathcal{F}(w)$.

Definition 5.2.24. If w is an ε-web, *a pseudo-colouring of w* is an application from the set of edges of w to the set $\mathcal{S}$ with no restriction. They are $3^{e(w)}$ different pseudo-colourings of w.

Proof of theorem 5.2.22. The global idea is to perform surgeries on every facet, this will lead to a big formal sum of foams and each of these foams will be geometrically a disjoint union of thetas-foam and of $l(\varepsilon)$ cups. Then we relate the colourings with the distributions of dots on these thetas and these cups.

Let us consider $\widehat{\mathrm{id}_w}$ as $\mathbb{S}^1 \times w$, and Δ the revolution axis of this foam. All the facets of $\widehat{\mathrm{id}_w}$ are diffeomorphic to a cylinder $I \times \mathbb{S}^1$. So we may perform surgeries on each of them. The surgery relation is not an embedded relation, so we could, considering $\widehat{\mathrm{id}_w}$ as a "non-closed pre-foam", apply this relation abstractly and get a big sum. However, we can perform everything in an embedded way. Furthermore, this is helpful for the sequel to

[3] *A priori* this is up to a sign but this is actually quite easy to check that if w is a closed web, the colourings of w have an even numbers of negative vertices if and only if w has an even number of edges.

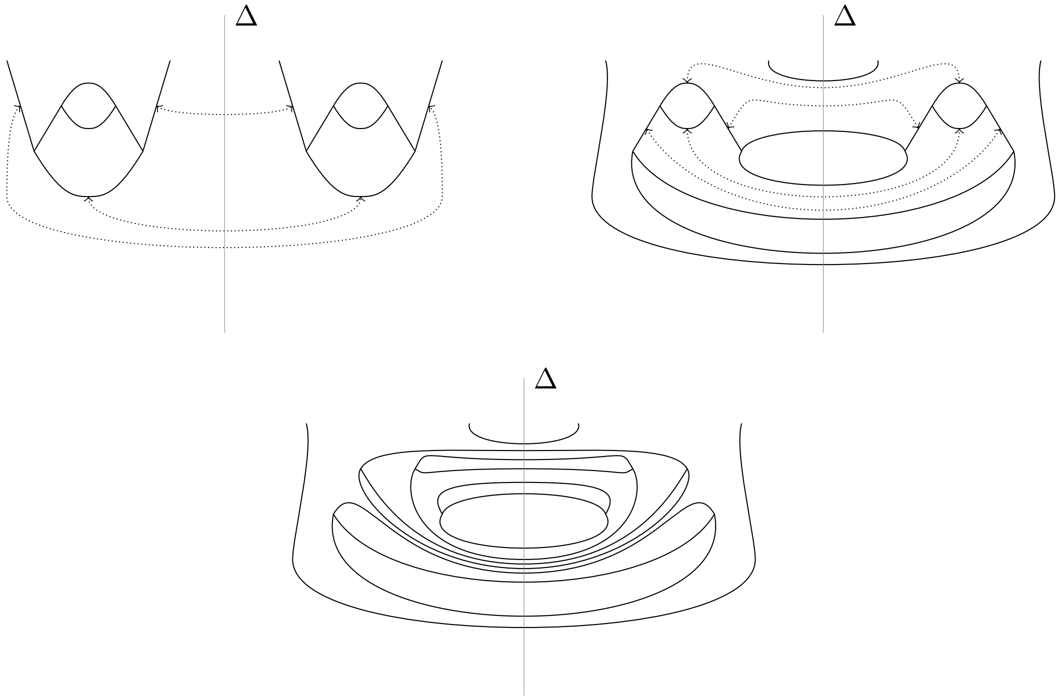


Figure 5.9: Cutted view of $\mathbb{S}^1 \times w$. We show here how to perform surgeries on all the facets of $\mathbb{S}^1 \times w$, the dotted two-side arrows indicates where the surgery takes place. It's clear that at the end we have only cups (represented here as open arcs) and theta foam represented here as theta graphs.

keep everything embedded. To do this, the only thing to do is to start with facets as close as possible from Δ and then go on inductively (see figure 5.9).

By "perform a surgery on a cylinder" we mean that we replace one foam which contains a cylinder by the linear combination given of figure 1.9. What we get in the end is a sum with $3^{e(w)}$ terms (each surgery multiply by three the number of terms) with an overall sign (each surgery relation contribute by -1). Each terms is a disjoint union of thetas foams (one for each cubic vertex of w) and of cups, one for each mono-valent vertex of w We call these terms resolutions of $\mathbb{S}^1 \times w$. One should now evaluate the thetas to understand which element of $A_{l(\varepsilon)}$ we get.

All facets of $\mathbb{S}^1 \times w$ corresponds to edges of w, so we may encode the different terms of our sum directly on the web w. We consider ϕ the application between pseudo-colourings of w and resolutions of $\mathbb{S}^1 \times w$. The colour of each vertex indicate which terms in the surgery relation we consider (see figure 5.10 for details). It's clear that all the pseudo colourings

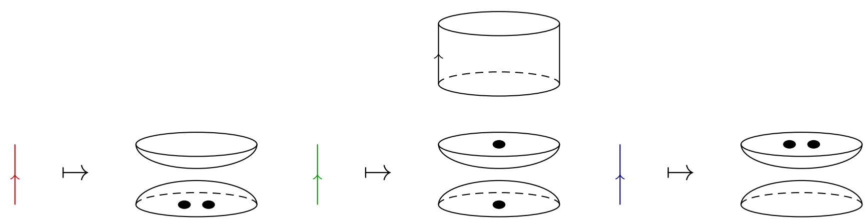


Figure 5.10: Description of the map ϕ.

which are not colourings are sent to resolutions which are zero because they contains some theta foams which doesn't have a good repartition of dots. It remains to understand the image by ϕ of a real colouring. A positive vertex of a colouring leads to "positive theta" *i.e.* theta-foam which evaluates on 1, while a negative vertex leads to a theta-foam which evaluates on -1. The application t is the right one to understand the distribution of dots on the cups. So for each colouring c of w the evaluation of the thetas of the corresponding resolution gives a $(-1)^{n_-^w(c)}$ factor, while the dots on the cups corresponds to a $P(\varepsilon, c_{|\varepsilon})$. This gives us the expected formula. $\square$

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